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A second-order parameter-uniform overlapping Schwarz method for reaction–diffusion problems with boundary layers

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Abstract

The problem of constructing a parameter-uniform numerical method for a singularly perturbed self-adjoint ordinary differential equation is considered. It is shown that a suitably designed discrete Schwarz method, based on a standard finite difference operator with a uniform mesh on each subdomain, gives numerical approximations which converge in the maximum norm to the exact solution, uniformly with respect to the singular perturbation parameter. This parameter-uniform convergence is shown to be essentially second order. That this new discrete Schwarz method is efficient in practice is demonstrated by numerical experiments.

Keywords: Singularly perturbed; Reaction–diffusion; Schwarz; Parameter-uniform

1. Introduction

On $\Omega=(0,1)$, we consider the following class of singularly perturbed reaction–diffusion problems:

$$L_\varepsilon u_\varepsilon(x) \equiv -\varepsilon u_\varepsilon''(x) + b(x)u_\varepsilon(x) = f(x), \quad x \in \Omega, \tag{1a}$$

$$u_\varepsilon(0) = A, \quad u_\varepsilon(1) = B, \tag{1b}$$

$$b(x) > \beta > 0 \quad \text{for all } x \in \bar{\Omega}, \tag{1c}$$

where the functions satisfy $b, f \in C^2(\bar{\Omega})$ and the singular perturbation parameter ε satisfies $0 < \varepsilon \leqslant 1$. For a singular perturbation problem of form (1), an appropriate norm for studying the convergence of numerical solutions to its exact solution is the maximum norm, which is defined by

$$\|\psi\|_{\bar{\Omega}} = \max_{x\in\bar{\Omega}} |\psi(x)|.$$

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The associated seminorms, defined for each integer $k \geqslant 0$, are

$$|\psi|_k = \|\psi^{(k)}\|.$$

Here we are interested in constructing an ε–uniform numerical method for this problem. By an ε- or parameter-uniform method, we mean a numerical method that generates numerical solutions $\{U_\varepsilon^N\}$ satisfying an ε-uniform error estimate of the form

$$\sup_{0<\varepsilon\leqslant 1} \|\bar{U}_\varepsilon^N - u_\varepsilon\|_{\bar{\Omega}} \leqslant CN^{-p}, \quad p > 0, \tag{2}$$

where C, p are constants independent of ε and N, and $\bar{U}_\varepsilon^N$ is the piecewise linear interpolant of the numerical solution U_ε^N. From (2), we see that we want to construct approximations that are defined at each point of the domain $\bar{\Omega}$, and not simply at the finite set of mesh points. We note also that in the definition of the norm the maximum is taken over the whole of $\bar{\Omega}$.

It is well known that classical numerical methods applied to singularly perturbed problems are not ε-uniform [3]. Therefore, various nonclassical approaches are used to design robust ε-uniform numerical methods for such problems. Such approaches can be either iterative or noniterative. With a noniterative approach it is necessary [3] to use a nonstandard finite difference operator (or a set of nonpolynomial basis functions in a finite element framework) and/or a specially distributed set of mesh or nodal points. On the other hand, with an iterative approach we can use standard Schwarz methods with overlapping subdomains (see, for example, [14] and the references therein), which in principle, permit the use of a different numerical method in each subdomain. This means that Schwarz methods can be adapted locally to any singularity in the solution that arises in a specific subdomain. A further advantage of such methods is that at each stage of the solution process it may be possible to avoid the use of a nonuniform mesh.

In the context of noniterative methods, parameter-uniform finite element methods with special exponential basis functions were examined for one-dimensional problems in [13], but the extension of these methods to partial differential equations is difficult. Sun and Stynes [16] showed that finite element methods based on standard piecewise polynomial basis functions and a piecewise-uniform fitted meshes are parameter-uniform in various norms for similar problems in one dimension. Finite difference methods, consisting of a centred finite difference operator on a piecewise-uniform fitted mesh, were also applied successfully to such problems (see, e.g. [10,11]) and extensions to higher dimensions were given in [15].

In the context of iterative methods, Garbey [6] and Garbey and Kaper [7] examined discrete Schwarz methods for singularly perturbed problems. In their methods the number of mesh points N is inversely proportional to the size of the singular perturbation parameter ε, which implies that these methods are not ε-uniform. Boglaev and Sirotkin [1] and Farrell et al. [4] examined Schwarz methods for singularly perturbed semi-linear analogues of problem (1), using a complicated fitted finite difference operator with special nonuniform meshes on the subdomains. In their methods strong restrictions on the distribution of the nodes are imposed that do not permit the use of a uniform mesh in each subdomain. Both Mathew [9] and Nataf and Rogier [12] examined the theoretical convergence properties of continuous, but not discrete, Schwarz methods for singularly perturbed problems. MacMullen et al. [8] showed by numerical experiments that for a one-dimensional singularly perturbed convection–diffusion problem, for which the continuous Schwarz method is known to be parameter-uniform, the corresponding discrete Schwarz method (using uniform meshes in each subdomain) is not parameter-uniform. Boglaev [2] examined a nonoverlapping Schwarz method for a time-dependent singularly perturbed analogue of problem (1), using a standard finite difference operator on a special piecewise-uniform mesh. However, this method is not ε-uniform, as the restriction $\varepsilon N \leqslant 1$ is imposed on the method.

In the present paper, a discrete Schwarz method based on a standard finite difference operator with a uniform mesh in each subdomain is constructed and shown to be ε-uniform for problem (1). This method is similar to the one proposed in Shishkin [15], where various theoretical results were announced. However, no detailed proofs, no consideration of numerical results and no iteration counts were provided in [15]. In the present paper, a detailed proof of the parameter-uniform convergence of the method is given and numerical results validating this theoretical result are presented. Iteration counts for the method are presented and their dependence on ε is also discussed.

An outline of the rest of the paper is as follows. In Section 2, the solution is decomposed into smooth and singular components. Parameter explicit bounds on the derivatives of these components are given. In Section 3, the continuous Schwarz method is introduced and error bounds are given. In Section 4, the discrete Schwarz method is described and bounds on the difference between the discrete Schwarz iterates and the continuous solution are derived. This leads to the main theoretical result of the paper: a parameter-uniform error estimate in the maximum norm of the discrete Schwarz iterates. In Section 5 results of a series of numerical experiments are presented which demonstrate the theoretical estimates derived in the earlier sections.

Notation. Throughout this paper, the letter C denotes a generic constant that is independent of the singular perturbation parameter ε, the discretization parameter N and the Schwarz iteration counter k.

2. The continuous problem

The reduced problem corresponding to (1) is the problem $b(x)v_0(x) = f(x)$ whose solution $v_0(x) = f(x)/b(x)$ cannot be made to satisfy arbitrary preassigned boundary conditions at the boundary points $\{0,1\}$ of Ω. Thus, in general, u_ε exhibits boundary layer behaviour at these points, the width of the boundary layers being $0(\sqrt{\varepsilon})$ (see, for example, [3] or [10]). It is well known that L_ε satisfies the following

Comparison principle: *Let $a,b \in \bar{\Omega}$ and assume that $\psi(a) \geqslant 0$ and $\psi(b) \geqslant 0$. Then $L_\varepsilon \psi(x) \geqslant 0$ for all $x \in (a,b)$ implies that $\psi(x) \geqslant 0$ for all $x \in [a,b] \subset \bar{\Omega}$.*

The uniqueness and stability of the solution u_ε of (1) are immediate consequences of this comparison principle.

We state without proof the following lemma which gives classical ε-explicit bounds on the derivatives of the solution of problems in a problem class containing problem (1). A proof of this lemma is given in [10].

Lemma 1. *Let ψ_ε be the solution of the problem*

$$L_\varepsilon \psi_\varepsilon = f, \quad x \in (a,b) \subset \Omega,$$

where $\psi_\varepsilon(a), \psi_\varepsilon(b)$ *are given and* $|\psi_\varepsilon(a)|, |\psi_\varepsilon(b)| \leqslant C$. *Then, for all* $k, 0 \leqslant k \leqslant 4$,

$$|\psi_\varepsilon^{(k)}(x)| \leqslant C(1 + \varepsilon^{-k/2}(\mathrm{e}^{-(x-a)\sqrt{\beta/\varepsilon}} + \mathrm{e}^{-(b-x)\sqrt{\beta/\varepsilon}})), \quad x \in [a, b],$$

where C *is a constant independent of* ε.

In what follows, we make extensive use of the following decomposition of the solution u_ε. We write $u_\varepsilon = v_\varepsilon + w_\ell + w_r$ where the smooth component v_ε and singular components w_ℓ, w_r are defined to be the solutions of the problems

$$L_\varepsilon v_\varepsilon = f, \quad v_\varepsilon(0) = f(0)/b(0), \quad v_\varepsilon(1) = f(1)/b(1),$$
$$L_\varepsilon w_\ell = 0, \quad w_\ell(0) = u_\varepsilon(0) - v_\varepsilon(0), \quad w_\ell(1) = 0,$$
$$L_\varepsilon w_r = 0, \quad w_r(0) = 0, \quad w_r(1) = u_\varepsilon(1) - v_\varepsilon(1).$$

This decomposition enables us to establish nonclassical sharper ε-explicit bounds on the derivatives of the solution of (1). These are contained in the following.

Lemma 2 (Miller et al. [11]). *The solution* u_ε *of* (1) *can be written in the form*

$$u_\varepsilon = v_\varepsilon + w_\ell + w_r,$$

where, for all k, $0 \leqslant k \leqslant 4$,

$$|v_\varepsilon|_k \leqslant C(1 + \varepsilon^{1-k/2})$$

and for all $x \in \bar{\Omega}$,

$$|w_\ell^{(k)}(x)| \leqslant C\varepsilon^{-k/2}\mathrm{e}^{-x\sqrt{\beta/\varepsilon}}, \qquad |w_r^{(k)}(x)| \leqslant C\varepsilon^{-k/2}\mathrm{e}^{-(1-x)\sqrt{\beta/\varepsilon}},$$

where C *is a constant independent of* ε.

3. Continuous Schwarz method

We now describe a continuous Schwarz method for (1), which is an iterative process generating a sequence of iterates $u_\varepsilon^{[k]}$, which converge as $k \to \infty$ to the exact solution u_ε. First, we introduce three overlapping subdomains of Ω:

$$\Omega_c = (\sigma, 1 - \sigma), \quad \Omega_l = (0, 2\sigma), \quad \Omega_r = (1 - 2\sigma, 1),$$

where the subdomain parameter σ is an appropriate constant, specified in Section 4, which satisfies

$$0 < \sigma \leqslant 0.25.$$

Then for each integer $k \geqslant 0$, the Schwarz iterates $u_\varepsilon^{[k]}$ are then defined as follows. For $k = 0$ we put

$$u_\varepsilon^{[0]}(x) \equiv 0, \quad 0 < x < 1, \qquad u_\varepsilon^{[0]}(0) = u_\varepsilon(0), \qquad u_\varepsilon^{[0]}(1) = u_\varepsilon(1)$$

and for all $k \geqslant 1$,

$$u_\varepsilon^{[k]} = \begin{cases} u_c^{[k]} & \text{in } \bar{\Omega}_c, \\ u_i^{[k]} & \text{in } \bar{\Omega}_i \backslash \Omega_c, \ i = l, r, \end{cases}$$

where the $u_i^{[k]}$ are the solutions of the problems

$$L_\varepsilon u_i^{[k]} = f \quad \text{in } \Omega_i, \qquad u_i^{[k]} = u_\varepsilon^{[k-1]} \quad \text{on } \partial\Omega_i, \ i = l, r$$

and

$$L_\varepsilon u_c^{[k]} = f \quad \text{in } \Omega_c, \qquad u_c^{[k]}(\sigma) = u_l^{[k]}(\sigma), \qquad u_c^{[k]}(1-\sigma) = u_r^{[k]}(1-\sigma).$$

The parameter-uniform convergence of these Schwarz iterates to u_ε is established in the following lemma. This is a well-known result (see, for example, [4,10,9].

Lemma 3. *For all* $k \geqslant 1$

$$\|u_\varepsilon^{[k]} - u_\varepsilon\|_{\bar\Omega} \leqslant Cq^k,$$

where C *is a constant independent of* k *and* ε *and*

$$q = \mathrm{e}^{-\sigma\sqrt{\beta/\varepsilon}} < 1.$$

4. Discrete Schwarz method

The discrete Schwarz method is obtained from the continuous Schwarz method by using a uniform mesh $\Omega_i^N, i = c, l, r$ on each subdomain Ω_i and replacing the differential operator L_ε by the standard centred finite difference operator L_ε^N. For any mesh function Z, on a uniform mesh with N subintervals L_ε^N is given by

$$L_\varepsilon^N Z_i = -\varepsilon\delta^2 Z_i + b(x_i)Z_i, \qquad \delta^2 Z_i = N^2(Z_{i+1} - 2Z_i + Z_{i-1}).$$

For each j, $0 \leqslant j \leqslant 2$, at each point of Ω_j associated with any mesh function Z defined on Ω_j^N, we define the piecewise linear interpolant $\bar Z_j$.

Then the sequence of discrete Schwarz iterates $\bar U_\varepsilon^{[k]}$ is defined by

$$\bar U_\varepsilon^{[0]}(x) \equiv 0, \quad 0 < x < 1, \qquad U_\varepsilon^{[0]}(0) = u_\varepsilon(0), \qquad U_\varepsilon^{[0]}(1) = u_\varepsilon(1).$$

For $k \geqslant 1$ the iterates $\bar U_\varepsilon^{[k]}$ are defined by

$$\bar U_\varepsilon^{[k]} = \begin{cases} \bar U_c^{[k]} & \text{in } \bar\Omega_c, \\ \bar U_i^{[k]} & \text{in } \bar\Omega_i \setminus \Omega_c, \ i = l, r, \end{cases}$$

where the $\bar U_i^{[k]}$ are the solutions of the problems

$$L_\varepsilon^N U_i^{[k]} = f \quad \text{in } \Omega_i^N, \qquad U_i^{[k]} = \bar U_\varepsilon^{[k-1]} \quad \text{on } \partial\Omega_i^N, \ i = l, r,$$

$$L_\varepsilon^N U_c^{[k]} = f \quad \text{in } \Omega_c^N, \qquad U_c^{[k]}(\sigma) = \bar U_l^{[k]}(\sigma), \qquad U_c^{[k]}(1-\sigma) = \bar U_r^{[k]}(1-\sigma),$$

and $\bar U$ is the linear interpolant of U.

Note that the centred finite difference operator L_ε^N satisfies the following discrete comparison principle in $\bar\Omega_j^N$, $j = l, c, r$.

Discrete comparison principle: *Assume that* $\Psi_0 \geqslant 0$ *and* $\Psi_N \geqslant 0$. *Then* $L_\varepsilon^N \Psi_i \geqslant 0$ *for all* $x_i \in \Omega_j^N$, $j = l, c, r$, *implies that* $\Psi_i \geqslant 0$ *for all* $x_i \in \bar\Omega_j^N$, $j = l, c, r$.

An immediate consequence of this is the following parameter-uniform stability result for L_ε^N. Let Z_i be any mesh function on Ω_j^N, $j = l, c, r$. Then for all i, $0 \leqslant i \leqslant N$,

$$|Z_i| \leqslant (1/\beta) \max_{1 \leqslant j \leqslant N-1} |L_\varepsilon^N Z_j| + \max\{Z_0, Z_N\}.$$

In order that the convergence properties of the discrete Schwarz method are parameter-uniform, we take the subdomain parameter σ to be

$$\sigma = \min\{\tfrac{1}{4}, 2\sqrt{\varepsilon/\beta} \ln N\}. \tag{3}$$

The discrete Schwarz iterates are now decomposed in an analogous way to u_ε. Thus we write

$$U_\varepsilon^{[k]} = V_\varepsilon^{[k]} + W_\ell^{[k]} + W_r^{[k]}.$$

Each term of $U_\varepsilon^{[k]}$ in the sequence of discrete Schwarz approximations is decomposed as follows:

$$\bar{U}_\varepsilon^{[k]} = \bar{V}_\varepsilon^{[k]} + \bar{W}_\ell^{[k]} + \bar{W}_r^{[k]} = \begin{cases} \bar{V}_c^{[k]} + \bar{W}_{\ell,c}^{[k]} + \bar{W}_{r,c}^{[k]} & \text{in } \bar{\Omega}_c, \\ \bar{V}_i^{[k]} + \bar{W}_{\ell,i}^{[k]} + \bar{W}_{r,i}^{[k]} & \text{in } \bar{\Omega}_i \setminus \Omega_c, \ i = l, r, \end{cases}$$

where

$$L_\varepsilon^N V_i^{[k]} = f \quad \text{in } \Omega_i^N, \qquad V_i^{[k]} = \bar{V}_\varepsilon^{[k-1]} \quad \text{on } \partial\Omega_i^N, \quad i = l, r,$$
$$L_\varepsilon^N V_c^{[k]} = f \quad \text{in } \Omega_c^N, \qquad V_c^{[k]}(\sigma) = \bar{V}_l^{[k]}(\sigma), \qquad V_c^{[k]}(1-\sigma) = \bar{V}_r^{[k]}(1-\sigma),$$

and for W_ℓ

$$L_\varepsilon^N W_{\ell,i}^{[k]} = 0 \quad \text{in } \Omega_i^N, \qquad W_{\ell,i}^{[k]} = \bar{W}_\ell^{[k-1]} \quad \text{on } \partial\Omega_i^N, \ \ i = l, r,$$
$$L_\varepsilon^N W_{\ell,c}^{[k]} = 0 \quad \text{in } \Omega_c^N, \qquad W_{\ell,c}^{[k]}(\sigma) = \bar{W}_{\ell,l}^{[k]}(\sigma), \qquad W_{\ell,c}^{[k]}(1-\sigma) = \bar{W}_{\ell,r}^{[k]}(1-\sigma),$$

and for W_r

$$L_\varepsilon^N W_{r,i}^{[k]} = 0 \quad \text{in } \Omega_i^N, \qquad W_{r,i}^{[k]} = \bar{W}_r^{[k-1]} \quad \text{on } \partial\Omega_i, \ \ i = l, r,$$
$$L_\varepsilon^N W_{r,c}^{[k]} = 0 \quad \text{in } \Omega_c^N, \qquad W_{r,c}^{[k]}(\sigma) = \bar{W}_{r,l}^{[k]}(\sigma), \qquad W_{r,c}^{[k]}(1-\sigma) = \bar{W}_{r,r}^{[k]}(1-\sigma).$$

The sequences are started by taking

$$\bar{V}_\varepsilon^{[0]}(x) \equiv 0, \quad 0 < x < 1, \qquad V_\varepsilon^{[0]}(0) = V_\varepsilon(0), \qquad V_\varepsilon^{[0]}(1) = V_\varepsilon(1),$$
$$\bar{W}_\ell^{[0]}(x) \equiv 0, \quad 0 < x < 1, \qquad W_\ell^{[0]}(0) = W_\ell(0), \qquad W_\ell^{[0]}(1) = 0,$$
$$\bar{W}_r^{[0]}(x) \equiv 0, \quad 0 < x < 1, \qquad W_r^{[0]}(0) = 0, \qquad W_r^{[0]}(1) = W_r(1).$$

In the following two lemmas parameter-uniform error estimates of the iterates are established. The first lemma concerns the smooth components.

Lemma 4. *Let v_ε and $\bar{V}^{[k]}$ denote the regular components of u_ε and $\bar{U}^{[k]}$, respectively, and let σ be chosen as in* (3). *Then, for all $k \geqslant 1$,*

$$\|v_\varepsilon - \bar{V}_\varepsilon^{[k]}\|_{\bar{\Omega}} \leqslant CN^{-2} + C2^{-k},$$

where C is a constant independent of k, N and ε.

Proof. Note that $(v_\varepsilon - V_l^{[1]})(0) = 0$ and $|(v_\varepsilon - V_l^{[1]})(2\sigma)| = |v_\varepsilon(2\sigma)| \leqslant C_0$. For $x_i \in \Omega_l^N$,

$$|L_\varepsilon^N(v_\varepsilon - V_l^{[1]})(x_i)| = |(L_\varepsilon^N - L_\varepsilon)(v_\varepsilon(x_i))| \leqslant C\varepsilon(2\sigma)^2N^{-2}|v_\varepsilon|_4 \leqslant C_1N^{-2}.$$

Here we have used the following standard local truncation error estimate for $z \in C^4(x_{i-1}, x_i)$ and $x_{i+1} - x_i = x_i - x_{i-1} = CN^{-1}$, then

$$|\delta^2 z - z''| \leqslant (CN)^{-2}|z|_{4,(x_{i-1},x_{i+1})}$$

and Lemma 2. Consider the two mesh functions

$$\frac{C_0x_i}{2\sigma} + \frac{C_1}{\beta}N^{-2} \pm (v_\varepsilon - V_l^{[1]})(x_i).$$

Then, from the discrete minimum principle, we get, for all $x_i \in \Omega_l^N$,

$$|(v_\varepsilon - V_l^{[1]})(x_i)| \leqslant \frac{C_0x_i}{2\sigma} + \frac{C_1}{\beta}N^{-2}.$$

Likewise, for all $x_i \in \Omega_r^N$,

$$|(v_\varepsilon - V_r^{[1]})(x_i)| \leqslant \frac{C_0(1-x_i)}{2\sigma} + \frac{C_1}{\beta}N^{-2}.$$

For all $x_i \in \Omega_c^N$, we obtain

$$|L_\varepsilon^N(v_\varepsilon - V_c^{[1]})(x_i)| = |(L_\varepsilon^N - L_\varepsilon)(v_\varepsilon(x_i))| \leqslant C\varepsilon N^{-2}|v_\varepsilon|_4 \leqslant C_1N^{-2},$$

with

$$|(v_\varepsilon - V_c^{[1]})(\sigma)| = |(v_\varepsilon - V_l^{[1]})(\sigma)| \leqslant \frac{C_0}{2} + C_1N^{-2}$$

and

$$|(v_\varepsilon - V_c^{[1]})(1-\sigma)| = |(v_\varepsilon - V_r^{[1]})(1-\sigma)| \leqslant \frac{C_0}{2} + C_1N^{-2}.$$

Hence

$$|(v_\varepsilon - V_c^{[1]})(x_i)| \leqslant C_1N^{-2} + \frac{C_0}{2}, \quad x_i \in \Omega_c^N.$$

Consider now the second iteration. Observe that $(v_\varepsilon - V_l^{[2]})(0)=0$ and $|(v_\varepsilon - V_l^{[2]})(2\sigma)| \leqslant C_1N^{-2} + C_0//2$. For $x_i \in \Omega_l^N$

$$|L_\varepsilon^N(v_\varepsilon - V_l^{[2]})(x_i)| \leqslant C_1N^{-2}.$$

Consider the two mesh functions

$$\left(\frac{C_0}{2}\right)\frac{x_i}{2\sigma} + \frac{C_1}{\beta}N^{-2} \pm (v_\varepsilon - V_l^{[2]})(x_i).$$

Then, from the discrete minimum principle, we get, for all $x_i \in \Omega_l^N$,

$$|(v_\varepsilon - V_l^{[2]})(x_i)| \leqslant \left(\frac{C_0}{2}\right)\frac{x_i}{2\sigma} + \frac{C_1}{\beta}N^{-2}$$

and for all $x_i \in \Omega_r^N$,

$$|(v_\varepsilon - V_r^{[2]})(x_i)| \leqslant \left(\frac{C_0}{2}\right)\frac{(1-x_i)}{2\sigma} + \frac{C_1}{\beta}N^{-2}.$$

Hence,

$$|(v_\varepsilon - V_c^{[2]})(\sigma)| \leqslant \frac{C_1}{\beta}N^{-2} + \frac{C_0}{4} \quad \text{and} \quad |(v_\varepsilon - V_c^{[2]})(1-\sigma)| \leqslant \frac{C_1}{\beta}N^{-2} + \frac{C_0}{4}$$

and thus

$$|(v_\varepsilon - V_c^{[2]})(x_i)| \leqslant \frac{C_1}{\beta}N^{-2} + \frac{C_0}{4}, \quad x_i \in \Omega_c^N.$$

We now use the standard interpolation error estimate for linear interpolation. That is, if $z \in C^2(x_{i-1}, x_i)$ and $\bar{z}$ is the linear interpolant then

$$\|z - \bar{z}\|_{\infty,(x_{i-1},x_i)} \leqslant C(x_i - x_{i-1})^2 \|z''\|_{\infty,(x_{i-1},x_i)}$$

which leads to

$$\begin{aligned}\|v_\varepsilon - \bar{V}_l^{[k]}\|_{\bar{\Omega}_l} &\leqslant \|\bar{V}_l^{[k]} - \bar{v}_\varepsilon\| + \|\bar{v}_\varepsilon - v_\varepsilon\| \\ &\leqslant \|\bar{V}_l^{[k]} - \bar{v}_\varepsilon\| + CN^{-2}(2\sigma)^2|v_\varepsilon|_2 \\ &\leqslant C_0 2^{-k} + \frac{C_1}{\beta}N^{-2} + C_2 N^{-2}.\end{aligned}$$

This completes the proof. □

The next lemma gives error estimates for the singular components.

Lemma 5. *Let w_ℓ, w_r and $\bar{W}_\ell^{[k]}$, $\bar{W}_r^{[k]}$ denote the singular components of u_ε and $\bar{U}^{[k]}$, respectively, and let σ be chosen as in* (3). *Then, for all $k \geqslant 1$ we have*
(i) $\|w_\ell - \bar{W}_\ell^{[k]}\|_{\bar{\Omega}} \leqslant C(N^{-1}\ln N)^2 + C2^{-k}$, (ii) $\|w_r - \bar{W}_r^{[k]}\|_{\bar{\Omega}} \leqslant C(N^{-1}\ln N)^2 + C2^{-k}$,
where C is a constant independent of k, N and ε.

Proof. We give the proof of (i); the proof of (ii) is analogous. Consider first the case when $\sigma < \frac{1}{4}$. For $x_i \in \Omega_l^N$,

$$\begin{aligned}|L_\varepsilon^N(w_\ell - W_{\ell,\ell}^{[1]})(x_i)| &= |(L_\varepsilon^N - L_\varepsilon)(w_\ell(x_i))| \leqslant C\varepsilon(2\sigma)^2 N^{-2}|w_l|_4 \\ &\leqslant C(2\sigma)^2 N^{-2}\varepsilon^{-1} \leqslant C(N^{-1}\ln N)^2.\end{aligned}$$

Hence, by the discrete minimum principle,

$$\|w_\ell - W_{\ell,\ell}^{[1]}\|_{\bar{\Omega}_l^N} \leqslant C(N^{-1}\ln N)^2.$$

Therefore

$$\|w_\ell - \bar{W}_{\ell,\ell}^{[1]}\|_{\bar{\Omega}_\ell} \leqslant C(N^{-1}\ln N)^2 + CN^{-2}(2\sigma)^2|w_\ell|_2 \leqslant C(N^{-1}\ln N)^2.$$

Likewise

$$\|w_\ell - \bar{W}_{\ell,r}^{[1]}\|_{\bar{\Omega}_r} \leqslant C(N^{-1}\ln N)^2.$$

Note that

$$L_\varepsilon^N(W_{\ell,c}^{[1]}) = 0, \quad \text{on } \Omega_c^N,$$

$$|W_{\ell,c}^{[1]}(\sigma)| = |\bar{W}_{\ell,l}^{[1]}(\sigma)| \leqslant |w_\ell(\sigma)| + C(N^{-1}\ln N)^2,$$

$$|W_{\ell,c}^{[1]}(1-\sigma)| = |\bar{W}_{\ell,r}^{[1]}(1-\sigma)| \leqslant |w_\ell(1-\sigma)| + C(N^{-1}\ln N)^2,$$

and we conclude that

$$|W_{\ell,c}^{[1]}|_{\bar{\Omega}_c^N} \leqslant C(N^{-1}\ln N)^2.$$

Hence

$$\|w_\ell - \bar{W}_{\ell,c}^{[1]}\|_{\bar{\Omega}_c} \leqslant \|w_\ell - \bar{w}_\ell\|_{\bar{\Omega}_c} + C(N^{-1}\ln N)^2.$$

Note that for any function z we have

$$|z - \bar{z}|_{(x_{i-1},x_i)} = \left| \int_{x_{i-1}}^{x} z'(t)\,\mathrm{d}t - \left(\int_{x_{i-1}}^{x_i} z'(t)\,\mathrm{d}t \right)(x - x_{i-1})/(x_i - x_{i-1}) \right| \leqslant \left| \int_{x_{i-1}}^{x_i} z'(t)\,\mathrm{d}t \right|,$$

and so, using Lemma 2, we have

$$\|w_\ell - \bar{w}_\ell\|_{(x_{i-1},x_i)} \leqslant \left| \int_{x_{i-1}}^{x_i} w_\ell'(t)\,\mathrm{d}t \right| \leqslant \mathrm{e}^{-x_{i-1}\sqrt{\beta/\varepsilon}} \leqslant \mathrm{e}^{-\sigma\sqrt{\beta/\varepsilon}} \leqslant C(N^{-1}\ln N)^2 \quad \text{for } x_{i-1} \geqslant \sigma.$$

We conclude that

$$\|w_\ell - \bar{W}_{\ell,c}^{[1]}\|_{\bar{\Omega}_c} \leqslant C(N^{-1}\ln N)^2.$$

The proof is completed by an induction argument. For the case of $\sigma = \frac{1}{4}$, use the argument in the previous lemma and note that $\sigma^2/\varepsilon \leqslant C(\ln N)^2$. □

Combining this with Lemma 3 immediately yields the main theoretical result of the paper, which is contained in

Theorem 1. *Let u_ε be the solution of problem* (1) *and let $\{\bar{U}_\varepsilon^{[k]}\}$ be the set of discrete Schwarz iterates with σ chosen as in* (3). *Then, for all $k \geqslant 1$*

$$\|u_\varepsilon - \bar{U}_\varepsilon^{[k]}\|_{\bar{\Omega}} \leqslant C(N^{-1}\ln N)^2 + C2^{-k},$$

where C is a constant independent of k, N and ε.

5. Numerical results

Numerical results are presented in this section, which confirm the theoretical estimates established in the previous section. The discrete Schwarz method described in Section 4 is applied to two problems from problem class (1). For notational reasons, it is helpful to introduce the piecewise-uniform mesh $\bar{\Omega}_\varepsilon^N$ associated with the overlapping subdomains by

$$\bar{\Omega}_\varepsilon^N \equiv \bar{\Omega}_c^N \cup (\bar{\Omega}_l^N \setminus \bar{\Omega}_c) \cup (\bar{\Omega}_r^N \setminus \bar{\Omega}_c). \tag{4}$$

For both examples, the stopping criterion for the Schwarz iterations is taken to be

$$\max_{x_i \in \bar{\Omega}_\varepsilon^N} |U_\varepsilon^{[k]}(x_i) - U_\varepsilon^{[k-1]}(x_i)| \leqslant 10^{-8}.$$

Our first problem is the constant coefficient problem

$$-\varepsilon u_\varepsilon''(x) + u_\varepsilon(x) = 0, \quad x \in \Omega, \tag{5a}$$

$$u_\varepsilon(0) = u_\varepsilon(1) = 1. \tag{5b}$$

Its exact solution in closed form is easy to find, which means that the exact pointwise errors can be calculated. The discrete Schwarz iterates are computed on a sequence of meshes with $N = 8, 16, \ldots, 1024$ for $\varepsilon = 2^{-2p}$, $p = 0, 1, 2, \ldots, 29$. Estimates of the global error

$$\|\bar{U}_\varepsilon^N - u_\varepsilon\|_{\bar{\Omega}}$$

are obtained by evaluating

$$E_{\varepsilon,\text{global}}^N = \max_{x_i \in \Omega^*} |\bar{U}_\varepsilon^N(x_i) - u_\varepsilon(x_i)|,$$

where $\Omega^* = \Omega_l^* \cup \Omega_c^* \cup \Omega_r^*$ and

$$\Omega_l^* = \{x_i \,|\, x_i = i\varepsilon/4096, 0 \leqslant i \leqslant 4096\},$$
$$\Omega_c^* = \{x_i \,|\, x_i = \varepsilon + i(1-2\varepsilon)/4096, 0 \leqslant i \leqslant 4096\},$$
$$\Omega_r^* = \{x_i \,|\, x_i = 1 - \varepsilon + i\varepsilon/4096, 0 \leqslant i \leqslant 4096\}.$$

Note that Ω^* depends on ε, but not on N.

Estimates of the parameter-uniform global pointwise error are obtained from

$$E_{\text{global}}^N = \max_\varepsilon E_{\varepsilon,\text{global}}^N$$

and estimates of the parameter-uniform order of convergence are computed for each N from

$$p_{\text{global}}^N = \log_2 \left(\frac{E_{\text{global}}^N}{E_{\text{global}}^{2N}} \right).$$

The values of $E_{\varepsilon,\text{global}}^N, E_{\text{global}}^N$ and p_{global}^N for the discrete Schwarz method applied to problem (5) are given in Table 1. In this and all subsequent tables, the dots indicate that the intermediate computed values are essentially the same as the given values. It is clear from Table 1 that this method is parameter-uniform for problem (5). The computed double-mesh order of convergence corresponding to an asymptotic convergence rate is $(N^{-1}\ln N)^2$ is

$$p_{\text{asym}}^N = \log_2 \left(\frac{(N^{-1}\ln N)^2}{(2N)^{-2}(\ln(2N))^2} \right) = 2\left(1 - \log_2\left(\frac{\ln 2N}{\ln N}\right)\right)$$

which correspond closely to the computed orders of convergence given in the last row of Table 1.

Table 1
Computed global maximum pointwise errors $E^N_{\varepsilon,\text{global}}$, E^N_{global} and parameter-uniform orders of convergence p^N_{global} for the discrete Schwarz method applied to problem (5) for various values of ε and N

ε	Number of intervals N in each subdomain							
	8	16	32	64	128	256	512	1024
2^{-0}	$1.90.10^{-3}$	$4.77.10^{-4}$	$1.19.10^{-4}$	$2.99.10^{-5}$	$7.47.10^{-6}$	$1.86.10^{-6}$	$4.59.10^{-7}$	$1.08.10^{-7}$
2^{-2}	$6.07.10^{-3}$	$1.55.10^{-3}$	$3.91.10^{-4}$	$9.83.10^{-5}$	$2.46.10^{-5}$	$6.16.10^{-6}$	$1.54.10^{-6}$	$3.81.10^{-7}$
2^{-4}	$1.42.10^{-2}$	$3.79.10^{-3}$	$9.71.10^{-4}$	$2.47.10^{-4}$	$6.21.10^{-5}$	$1.56.10^{-5}$	$3.90.10^{-6}$	$9.68.10^{-7}$
2^{-6}	$2.34.10^{-2}$	$6.82.10^{-3}$	$1.82.10^{-3}$	$4.72.10^{-4}$	$1.20.10^{-4}$	$3.01.10^{-5}$	$7.47.10^{-6}$	$1.90.10^{-6}$
2^{-8}	$2.61.10^{-2}$	$7.88.10^{-3}$	$2.28.10^{-3}$	$6.13.10^{-4}$	$1.58.10^{-4}$	$4.03.10^{-5}$	$1.01.10^{-5}$	$2.37.10^{-6}$
2^{-10}	$8.46.10^{-2}$	$2.60.10^{-2}$	$7.16.10^{-3}$	$1.87.10^{-3}$	$4.78.10^{-4}$	$1.21.10^{-4}$	$3.04.10^{-5}$	$7.61.10^{-6}$
2^{-12}	$8.99.10^{-2}$	$4.62.10^{-2}$	$2.00.10^{-2}$	$7.16.10^{-3}$	$1.87.10^{-3}$	$4.78.10^{-4}$	$1.21.10^{-4}$	$3.04.10^{-5}$
2^{-14}	$8.99.10^{-2}$	$4.62.10^{-2}$	$2.00.10^{-2}$	$7.70.10^{-3}$	$2.71.10^{-3}$	$9.02.10^{-4}$	$2.79.10^{-4}$	$8.80.10^{-5}$
2^{-16}	$8.99.10^{-2}$	$4.60.10^{-2}$	$1.98.10^{-2}$	$7.71.10^{-3}$	$2.69.10^{-3}$	$8.59.10^{-4}$	$2.62.10^{-4}$	$8.68.10^{-5}$
2^{-18}	$8.99.10^{-2}$	$4.57.10^{-2}$	$1.95.10^{-2}$	$7.70.10^{-3}$	$2.22.10^{-3}$	$8.55.10^{-4}$	$2.62.10^{-4}$	$7.29.10^{-5}$
2^{-20}	$8.99.10^{-2}$	$4.33.10^{-2}$	$1.95.10^{-2}$	$2.70.10^{-3}$	$2.24.10^{-3}$	$5.08.10^{-4}$	$1.74.10^{-4}$	$6.48.10^{-5}$
2^{-22}	$8.99.10^{-2}$	$3.75.10^{-2}$	$9.78.10^{-3}$	$2.72.10^{-3}$	$1.73.10^{-3}$	$4.96.10^{-4}$	$1.71.10^{-4}$	$6.48.10^{-5}$
2^{-24}	$2.49.10^{-2}$	$2.81.10^{-2}$	$9.78.10^{-3}$	$2.73.10^{-3}$	$1.38.10^{-3}$	$4.58.10^{-4}$	$1.45.10^{-4}$	$1.87.10^{-5}$
2^{-26}	$1.87.10^{-2}$	$9.21.10^{-3}$	$5.20.10^{-3}$	$1.76.10^{-3}$	$4.92.10^{-4}$	$1.23.10^{-4}$	$3.27.10^{-5}$	$1.39.10^{-5}$
2^{-28}	$1.87.10^{-2}$	$4.33.10^{-3}$	$1.01.10^{-3}$	$3.51.10^{-4}$	$1.19.10^{-4}$	$3.23.10^{-5}$	$8.73.10^{-6}$	$3.55.10^{-6}$
2^{-30}	$1.88.10^{-2}$	$4.34.10^{-3}$	$1.02.10^{-3}$	$2.46.10^{-4}$	$6.00.10^{-5}$	$1.46.10^{-5}$	$3.50.10^{-6}$	$7.78.10^{-7}$
2^{-32}	$1.88.10^{-2}$	$4.35.10^{-3}$	$1.03.10^{-3}$	$2.48.10^{-4}$	$6.07.10^{-5}$	$1.49.10^{-5}$	$3.60.10^{-6}$	$8.10.10^{-7}$
2^{-34}	$1.88.10^{-2}$	$4.35.10^{-3}$	$1.02.10^{-3}$	$2.47.10^{-4}$	$6.03.10^{-5}$	$1.47.10^{-5}$	$3.53.10^{-6}$	$8.19.10^{-7}$
2^{-36}	$1.88.10^{-2}$	$4.34.10^{-3}$	$1.02.10^{-3}$	$2.47.10^{-4}$	$6.00.10^{-5}$	$1.45.10^{-5}$	$3.44.10^{-6}$	$7.67.10^{-7}$
2^{-38}	$1.88.10^{-2}$	$4.34.10^{-3}$	$1.02.10^{-3}$	$2.46.10^{-4}$	$5.98.10^{-5}$	$1.44.10^{-5}$	$3.39.10^{-6}$	$7.41.10^{-7}$
2^{-40}	$1.88.10^{-2}$	$4.34.10^{-3}$	$1.02.10^{-3}$	$2.46.10^{-4}$	$5.98.10^{-5}$	$1.44.10^{-5}$	$3.37.10^{-6}$	$7.28.10^{-7}$
.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.
2^{-58}	$1.88.10^{-2}$	$4.34.10^{-3}$	$1.02.10^{-3}$	$2.46.10^{-4}$	$5.97.10^{-5}$	$1.44.10^{-5}$	$3.34.10^{-6}$	$7.16.10^{-7}$
E^N_{global}	$8.99.10^{-2}$	$4.62.10^{-2}$	$2.00.10^{-2}$	$7.71.10^{-3}$	$2.71.10^{-3}$	$9.02.10^{-4}$	$2.79.10^{-4}$	$8.80.10^{-5}$
p^N_{global}	$9.59.10^{-1}$	$1.21.10^{+0}$	$1.38.10^{+0}$	$1.51.10^{+0}$	$1.59.10^{+0}$	$1.69.10^{+0}$	$1.66.10^{+0}$	$1.67.10^{+0}$

Our second problem is the variable coefficient problem

$$-\varepsilon u''_\varepsilon(x) + (1+x^2)u_\varepsilon(x) = x^3, \quad x \in \Omega, \tag{6a}$$

$$u_\varepsilon(0) = u_\varepsilon(1) = 1. \tag{6b}$$

In this case, the exact solution is not used to estimate the numerical errors. Instead, the nodal errors and orders of convergence are estimated using the double-mesh principle modified in accordance with the parameter-robust definition (see [5] for the nodal double mesh principle). The double-mesh differences are defined by

$$D^N_\varepsilon \equiv \max_{x_i \in \bar{\Omega}^N_\varepsilon} |U^N_\varepsilon(x_i) - \bar{U}^{2N}_\varepsilon(x_i)|$$

Table 2
Computed maximum pointwise errors E_ε^N, E^N and parameter-uniform order of convergence p^N for the discrete Schwarz method applied to problem (6) for various values of ε and N

ε	Number of intervals N in each subdomain							
	8	16	32	64	128	256	512	1024
2^{-0}	$9.01.10^{-4}$	$2.25.10^{-4}$	$5.62.10^{-5}$	$1.41.10^{-5}$	$3.51.10^{-6}$	$8.75.10^{-7}$	$2.16.10^{-7}$	$5.14.10^{-8}$
2^{-2}	$2.58.10^{-3}$	$6.52.10^{-4}$	$1.62.10^{-4}$	$4.06.10^{-5}$	$1.01.10^{-5}$	$2.53.10^{-6}$	$6.24.10^{-7}$	$1.48.10^{-7}$
2^{-4}	$4.83.10^{-3}$	$1.25.10^{-3}$	$3.07.10^{-4}$	$7.73.10^{-5}$	$1.92.10^{-5}$	$4.80.10^{-6}$	$1.19.10^{-6}$	$2.82.10^{-7}$
2^{-6}	$5.18.10^{-3}$	$1.47.10^{-3}$	$3.60.10^{-4}$	$9.10.10^{-5}$	$2.26.10^{-5}$	$5.66.10^{-6}$	$1.40.10^{-6}$	$3.33.10^{-7}$
2^{-8}	$4.42.10^{-3}$	$1.51.10^{-3}$	$3.85.10^{-4}$	$9.71.10^{-5}$	$2.43.10^{-5}$	$6.07.10^{-6}$	$1.50.10^{-6}$	$3.57.10^{-7}$
2^{-10}	$1.41.10^{-2}$	$3.74.10^{-3}$	$9.71.10^{-4}$	$2.45.10^{-4}$	$6.14.10^{-5}$	$1.53.10^{-5}$	$3.78.10^{-6}$	$9.01.10^{-7}$
2^{-12}	$1.51.10^{-2}$	$6.87.10^{-3}$	$2.80.10^{-3}$	$9.58.10^{-4}$	$2.41.10^{-4}$	$6.04.10^{-5}$	$1.49.10^{-5}$	$3.56.10^{-6}$
2^{-14}	$1.51.10^{-2}$	$6.82.10^{-3}$	$2.80.10^{-3}$	$1.03.10^{-3}$	$3.51.10^{-4}$	$1.14.10^{-4}$	$3.54.10^{-5}$	$1.00.10^{-5}$
2^{-16}	$1.51.10^{-2}$	$6.80.10^{-3}$	$2.80.10^{-3}$	$1.03.10^{-3}$	$3.49.10^{-4}$	$1.14.10^{-4}$	$3.55.10^{-5}$	$1.03.10^{-5}$
2^{-18}	$1.51.10^{-2}$	$6.79.10^{-3}$	$2.80.10^{-3}$	$1.03.10^{-3}$	$3.49.10^{-4}$	$1.14.10^{-4}$	$3.54.10^{-5}$	$1.02.10^{-5}$
2^{-20}	$1.51.10^{-2}$	$6.79.10^{-3}$	$2.80.10^{-3}$	$1.03.10^{-3}$	$3.49.10^{-4}$	$1.14.10^{-4}$	$3.54.10^{-5}$	$1.02.10^{-5}$
.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.
2^{-58}	$1.51.10^{-2}$	$6.78.10^{-3}$	$2.80.10^{-3}$	$1.03.10^{-3}$	$3.49.10^{-4}$	$1.14.10^{-4}$	$3.54.10^{-5}$	$1.02.10^{-5}$
E^N_{nodal}	$1.51.10^{-2}$	$6.87.10^{-3}$	$2.80.10^{-3}$	$1.03.10^{-3}$	$3.51.10^{-4}$	$1.14.10^{-4}$	$3.55.10^{-5}$	$1.03.10^{-5}$
p^N	$7.81.10^{-1}$	$1.06.10^{+0}$	$1.29.10^{+0}$	$1.43.10^{+0}$	$1.35.10^{+0}$	$1.55.10^{+0}$	$1.64.10^{+0}$	$1.64.10^{+0}$

and the parameter-uniform differences are defined by

$$D^N = \max_\varepsilon D_\varepsilon^N.$$

From these the parameter-uniform order of convergence is computed from

$$p^N = \log_2\left(\frac{D^N}{D^{2N}}\right).$$

The numerical errors are then estimated by using the Schwarz solution on the finest available mesh, corresponding to $N=4096$, as an approximation to the exact solution in the expression for the error. The corresponding computed maximum pointwise error is taken to be

$$E^N_{\varepsilon,\mathrm{nodal}} = \max_{x_i \in \bar{\Omega}_\varepsilon^N} |U_\varepsilon^N(x_i) - \bar{U}_\varepsilon^{4096}(x_i)|$$

and for each N we define the computed parameter-uniform pointwise error by

$$E^N_{\mathrm{nodal}} = \max_\varepsilon E^N_{\varepsilon,\mathrm{nodal}}.$$

Values of $E^N_{\varepsilon,\mathrm{nodal}}, E^N_{\mathrm{nodal}}$ and p^N for the discrete Schwarz method applied to problem (6) are given in Table 2. They show experimentally that the method is parameter-uniform for problem (6). Iteration counts for various values of ε and N for the discrete Schwarz method applied to problem (6) are

Table 3
Iteration count for the discrete Schwarz method applied to problem (6) for various values of ε and N

ε	Number of intervals N in each subdomain							
	8	16	32	64	128	256	512	1024
2^{-0}	25	25	25	25	25	25	25	25
2^{-2}	21	21	21	21	21	21	21	21
2^{-4}	15	15	15	15	15	15	15	15
2^{-6}	10	10	10	10	10	10	10	10
2^{-8}	6	6	6	6	6	6	6	6
2^{-10}	4	4	4	4	4	4	4	4
2^{-12}	3	3	3	3	3	3	3	3
2^{-14}	4	3	3	3	3	3	2	2
.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.
2^{-58}	4	3	3	3	3	3	2	2

given in Table 3. We see that these iteration counts are essentially independent of N and decrease with decreasing values of ε.

6. Conclusions

In this paper, a one-dimensional singularly perturbed reaction–diffusion problem was examined. It was shown that a suitably designed discrete Schwarz method gives approximations which converge in the maximum norm to the exact solution uniformly with respect to the singular perturbation parameter. This parameter-uniform convergence was shown to be essentially second order. Numerical results were presented, which show that, for a small value of the parameter ε, only a few iterations are required and that the number of iterations is independent of the number of mesh points used. The method, considered here for the one-dimensional reaction–diffusion equation, can be extended to multi-dimensional problems.

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