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V.V. Arestov and A.G. Babenko, On Kissing Number in Four Dimensions

Let τ_m denote the greatest number of nonoverlapping equal balls that touch another ball of the same radius in $\mathbf{R}^m$. Due to A.M. Odlyzko and N.J.A. Sloane [11] the problem of determination of the number τ_4 (kissing number problem in $\mathbf{R}^4$) was reduced to investigation of minimal distances of arrangements of 25 points on unit sphere $\mathbf{S}^3$ in $\mathbf{R}^4$. In this presentation we demonstrate possibilities of the classical Delsarte method for estimating the upper bounds of minimal distances of arrangements of 25 and 24 points on $\mathbf{S}^3$.

Introduction. Let $\mathbf{S}^{m-1}$ denote the unit sphere in real Euclidean m-dimensional space $\mathbf{R}^m$: $\mathbf{S}^{m-1} = \{x \in \mathbf{R}^m : \ xx = 1\}$, where $xy = x_1y_1 + x_2y_2 + \ldots + x_my_m$ is the inner product of vectors (points) $x = (x_1, x_2, \ldots, x_m)$, $y = (y_1, y_2, \ldots, y_m) \in \mathbf{R}^m$. For $-1 \le s < 1$ a set $W \subset \mathbf{S}^{m-1}$ containing at least two points is called *a spherical s-code,* if for any two points $x, y \in W$, $x \ne y$, their inner product xy satisfies to the condition $xy \le s$. This condition means that an angular distance (angle) between any pair of distinct points in W is greater than or equal to $\arccos s$:

$$\widetilde{\varphi}(W) := \min\{\arccos(xy) : \ x \ne y, \ x, y \in W\} \ge \arccos s.$$

The *size* of W is simply the number of points in the set W; we denote it by $|W|$.

The basic problem for spherical codes can be formulated as follows: for given $m \ge 2$ and $-1 \le s < 1$, find a spherical s-code $W \subset \mathbf{S}^{m-1}$ of largest possible size $|W|$. We denote this maximal size by $M(m, s)$. This problem can be consider in the inverse form: for given $m \ge 2$ and $N \ge 2$, find a spherical code $W \subset \mathbf{S}^{m-1}$, $|W| = N$ with largest possible minimal angle $\widetilde{\varphi}(W)$, which we denote by $\varphi_m(N)$, i.e.

$$\varphi_m(N) = \max\{\, \widetilde{\varphi}(W) : \ W \subset \mathbf{S}^{m-1}, \quad |W| = N \,\}.$$

The indicated problem and the kissing number problem are connected by equality (cf. [3])

$$\tau_m = M(m, 1/2), \quad m \ge 2. \tag{1}$$

The known example (cf. [3, Ch. 1, § 2]) of spherical $1/2$-code in $\mathbf{R}^4$ is the set consisting from 24 vertexes of the regular convex polyhedron with Schläfli symbol $\{3, 4, 3\}$; therefore $24 \le \tau_4$. Odlyzko and Sloane [11] have proved that τ_4 does not exceed 25; thus,

$$24 \le \tau_4 \le 25. \tag{2}$$

In the articles [9], [10], Levenstein obtained universal upper bound for the quantity $M(m,s)$. Levenstein's result can be reformulated for estimating $\varphi_m(N)$ (cf. [2]); in particular,

$$\varphi_4(25) < 60.79^\circ, \quad \varphi_4(24) < 61.65^\circ.$$

For concrete cases stronger estimations were presented by Boyvalenkov, Danev and Bumova [2]; for example, (in our notations) they obtained the inequality

$$\varphi_4(24) < 61.47^\circ.$$

The arrangement of 25 points on $\mathbf{S}^3$ with the minimal angle 57.4988826° was constructed by Hardin, Sloane and Smith [7]. Thus, the following estimations are fulfilled:

$$57.4988826^\circ \le \varphi_4(25) < 60.79^\circ, \quad 60^\circ \le \varphi_4(24) < 61.47^\circ.$$

Delsarte method. The method which arose in investigations of Delsarte [4] on upper bounds for packings in some metric spaces is often used for estimating from above the quantities $M(m,s),\ \varphi_m(N)$. This method was developed and successful applied in the works [5], [8], [11], [9], [10], [2] and in of many other articles; the rich bibliography on the subject is contained in monograph [3].

Let $R_k = R_k^{\alpha,\alpha},\ \ k = 0,1,2,\dots,$ be the system of ultraspherical (Gegenbauer) polynomials which are orthogonal on interval $[-1,1]$ with respect to weight function $v(t) = (1-t^2)^\alpha, \quad \alpha = (m-3)/2$, and normed by condition $R_k(1) = 1$. The mentioned approach is founded on a positive-definite property of these polynomials (if we will consider them as kernels $R_k(xy)$ on on the Cartesian product $\mathbf{S}^{m-1} \times \mathbf{S}^{m-1}$).

For $-1 \le s < 1,\ m \ge 2$, denote by $\mathcal{F}_m(s)$ a set consisting of all continuous functions f on $[-1,1]$ which are non-positive on $[-1,s]: \quad f(t) \le 0,\ t \in [-1,s]$, and represented on $[-1,1]$ by series $f(t) = f_0R_0(t) + f_1R_1(t) + f_2R_2(t) + \dots$ with non-negative coefficients $f_k \ge 0,\ k \ge 1$, and $f_0 > 0$. The fact that $\mathcal{F}_m(s)$ is non-empty for all $-1 \le s < 1,\ m \ge 2$ was proved in [8]. Let us put

$$w_m(s) = \inf\left\{\frac{f(1)}{f_0} :\ f \in \mathcal{F}_m(s)\right\}, \quad -1 \le s < 1,\ m \ge 2.$$

The following statement is contained in [5], [8].

Theorem A. *Let* $s \in [-1,1), \quad m \ge 2.$ *Then* $M(m,s) \le w_m(s).$

Formulation of results. Further we consider four-dimensional case ($m = 4$) only. In this case index α is equal to $1/2$ and the polynomials $R_k = R_k^{1/2,1/2}$ are Chebyshev polynomials of the second kind:

$$R_k(t) = \frac{\sin(k+1)\theta}{(k+1)\sin\theta}, \quad t = \cos\theta, \quad k = 0, 1, 2, \ldots$$

Using technique worked out by authors in [1] we find exact values of a parameter s for which function $w_4(s)$ takes values 25 and 24. For formulating the result, we need the following polynomial:

$$\begin{aligned} H(z) &= 1744568320000\, z^{28} + 19824640000\, z^{27} - 11368270848000\, z^{26} + \\ &+ 299992125440\, z^{25} + 33683617005056\, z^{24} - 1690611799808\, z^{23} - \\ &- 59756580346080\, z^{22} + 3740858012128\, z^{21} + 70524254066704\, z^{20} - \\ &- 4516619739088\, z^{19} - 58188563861056\, z^{18} + 3200479271680\, z^{17} + \\ &+ 34328475907496\, z^{16} - 1262955136312\, z^{15} - 14563330120710\, z^{14} + \\ &+ 172742066070\, z^{13} + 4417415566665\, z^{12} + 76811504675\, z^{11} - \\ &- 942777154875\, z^{10} - 46753060057\, z^{9} + 137285137301\, z^{8} + \\ &+ 11621133345\, z^{7} - 12856584451\, z^{6} - 1594636173\, z^{5} + 680106134\, z^{4} + \\ &+ 118057108\, z^{3} - 13255560\, z^{2} - 3691008\, z - 186624. \end{aligned}$$

Theorem 1. *In $[-1, 1)$ the equation $w_4(s) = 25$ has the unique solution: $s = z_{13}$, where $z_{13} = 0.4925150241\ldots$ is thirteenth $($on the increase$)$ real root of polynomial H.*

Consider the polynomial

$$\begin{aligned} h(z) &= 6068404224\, z^{24} - 5559746560\, z^{23} - 32435331072\, z^{22} + 30162632704\, z^{21} + \\ &+ 76657888256\, z^{20} - 69950994432\, z^{19} - 105547058176\, z^{18} + 90905438208\, z^{17} + \\ &+ 93805633312\, z^{16} - 72899067584\, z^{15} - 56276296952\, z^{14} + 37463407248\, z^{13} + \\ &+ 23144486195\, z^{12} - 12425086062\, z^{11} - 6505367271\, z^{10} + 2613609108\, z^{9} + \\ &+ 1227229561\, z^{8} - 331172622\, z^{7} - 149222121\, z^{6} + 22205608\, z^{5} + \\ &+ 10721860\, z^{4} - 490544\, z^{3} - 368104\, z^{2} - 10880\, z + 2432. \end{aligned}$$

Theorem 2. *In $[-1, 1)$ the equation $w_4(s) = 24$ has the unique solution: $s = x_{12}$, where $x_{12} = 0.4785451836\ldots$ is twelfth $($on the increase$)$ real root of polynomial h.*

Corollary. *Among arbitrary* 25 $($*resp.* $24)$ *points located on* $\mathbf{S}^3 \subset \mathbf{R}^4$ *there exist two points with the angle between them strictly less than* 60.5° $($*resp.* $61.41^\circ)$. *Thus,*

$$\varphi_4(25) < 60.5^\circ, \quad \varphi_4(24) < 61.41^\circ.$$

Let us remind also (see (1), (2)) that the answer to the question: "whether is the least angular distance between pairs of points of arbitrary configuration of 25 points on $\mathbf{S}^3$ strictly less than 60 degrees?" should give a solution of the kissing number problem in $\mathbf{R}^4$.

The theorem 1 implies that in order to find the number τ_4 it is necessary to use other methods. We hope that application of results and ideas of Paul Erdös, his coauthors and pupils, in particular, concerning properties of distribution of distances of given number points on sphere [6], will help to decide problems of such kind.

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