History

of

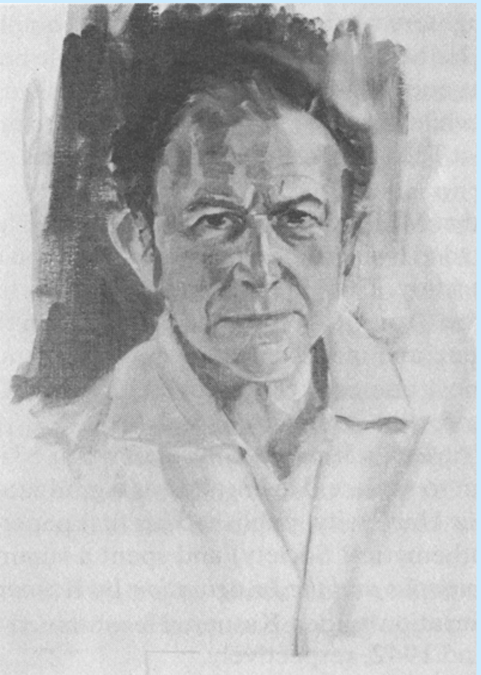
the Homicidal Chauffeur Problem

Dedicated to

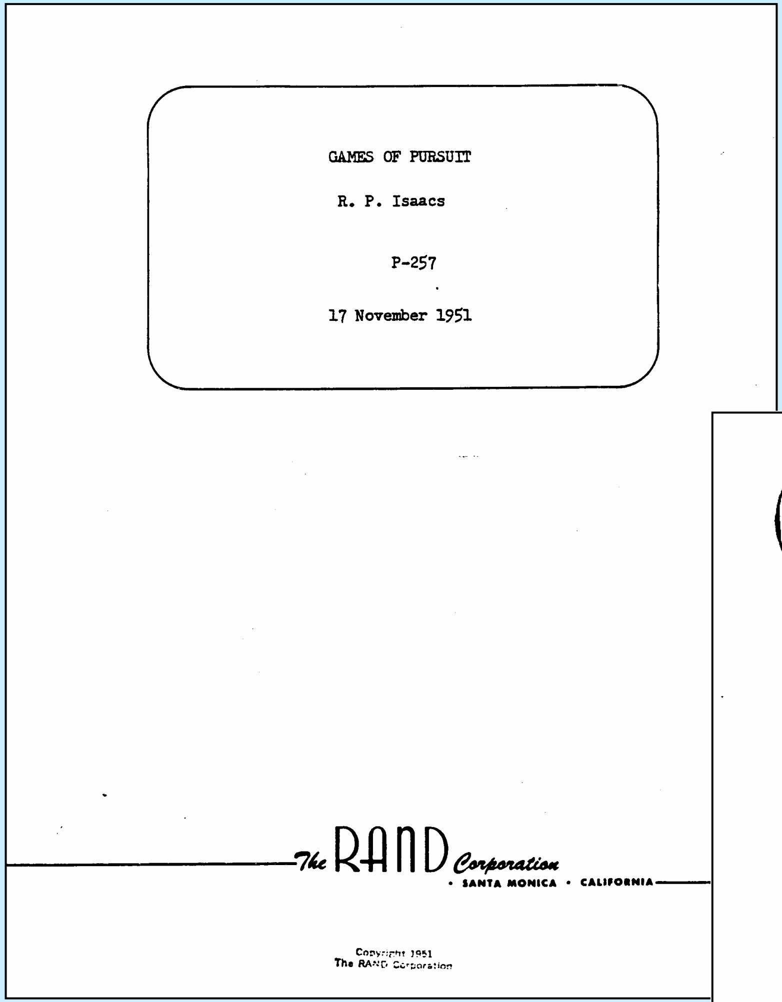
Pierre Bernhard

on his 60th birthday

Statement of the problem by R.Isaacs in the early fifties



R.Isaacs



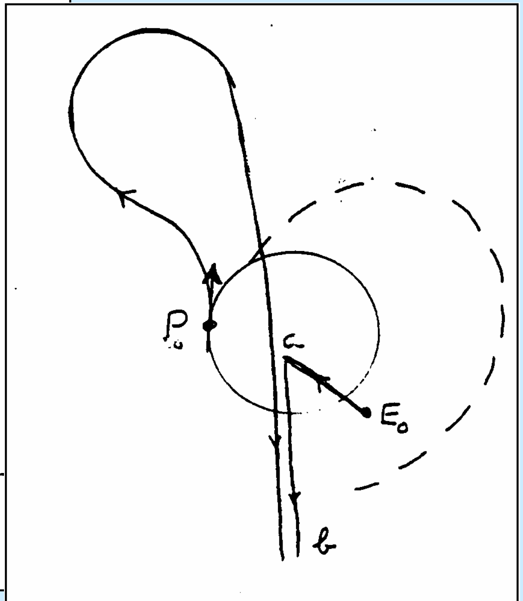


Fig. 1

Rufus Isaacs has formulated his famous problem in 1951. The problem was published in a report of Rand Corporation. The principal steps toward the solution were also outlined in this paper.

Isaacs´ book. Dynamics of the homicidal chauffeur game

P: $\dot{x}_p = w^{(1)} \sin \psi$

$\dot{y}_p = w^{(1)} \cos \psi$

$\dot{\psi} = w^{(1)} \varphi / R$

$|\varphi| \leq 1, \quad w^{(1)} = \text{const} > 0$

$R = \text{const} > 0$

E: $\dot{x}_e = v_1$

$\dot{y}_e = v_2$

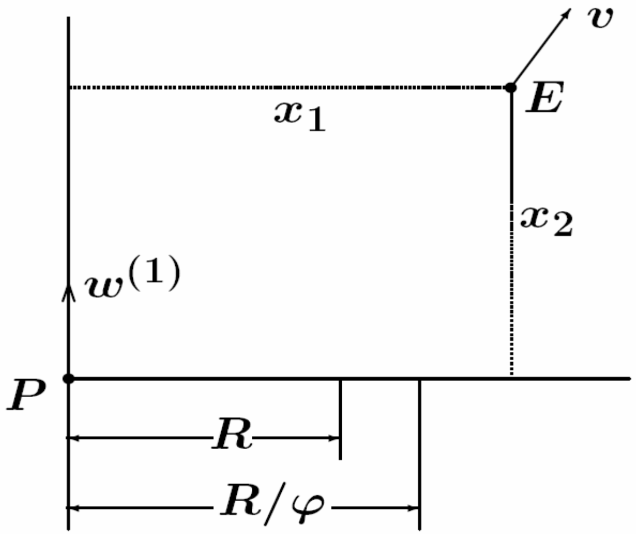
$v = (v_1, v_2)', \quad |v| \leq w^{(2)}$

In reduced coordinates

$$\dot{x}_1 = -w^{(1)} x_2 \varphi / R + v_1$$
$$\dot{x}_2 = w^{(1)} x_1 \varphi / R + v_2 - w^{(1)}$$

$$|\varphi| \leq 1, \; |v| \leq w^{(2)}$$

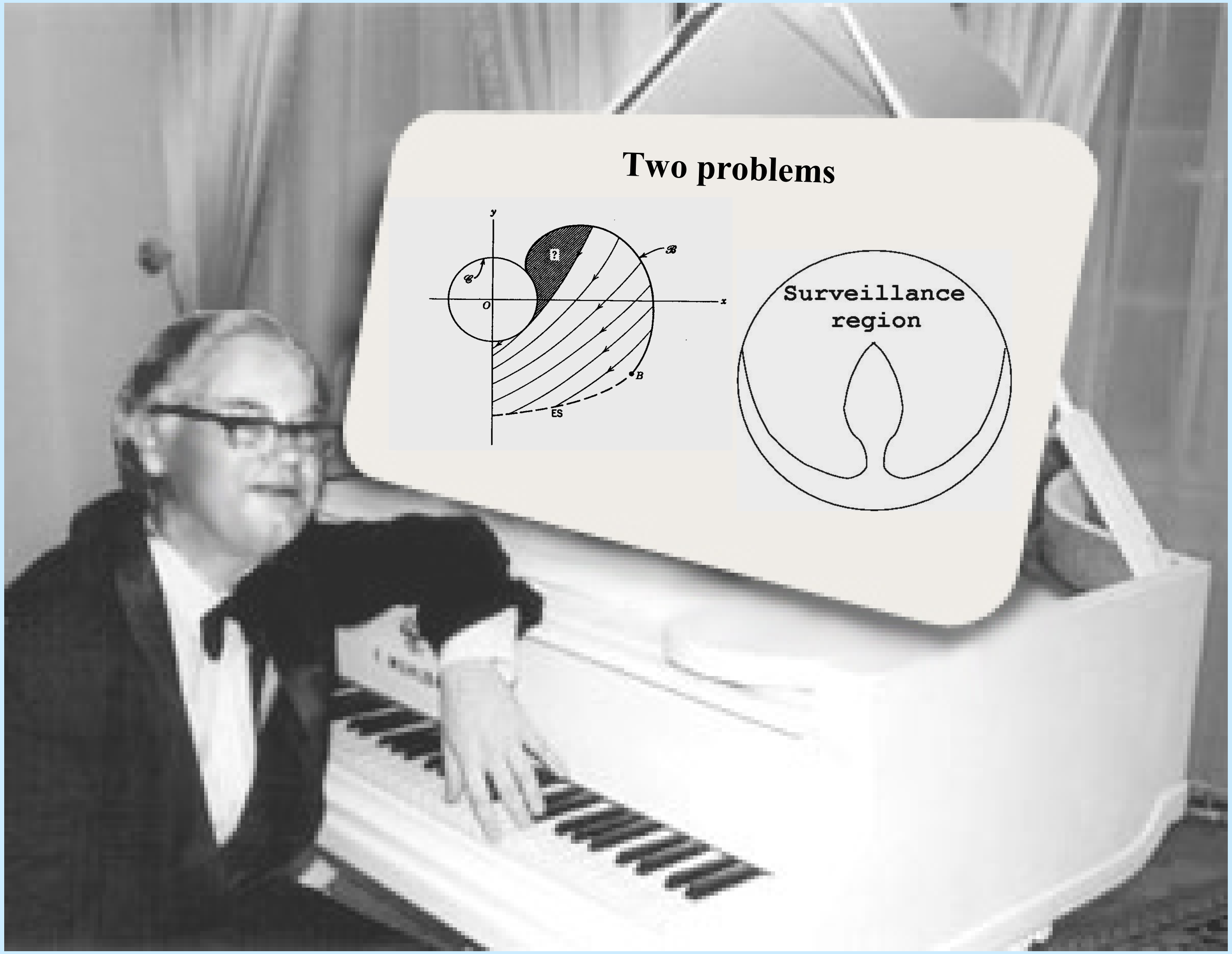
Classical game: the goal of player P is to bring the state x to the terminal circle as soon as possible



The description of the dynamics is given here on the top. Object P possesses the dynamics of a car. Object E is an inertialess pedestrian that is able to change the direction of his velocity instantaneously. Such a description turned out to be very appropriate for many practical problems: P is an inertial aircraft, E is more manoeuvrable one.

The dynamics in reduced coordinates is given below. The vertical axis x_2 is directed along the velocity vector of player P, the axis x_1 is orthogonal to x_2. The origin coincides with the current position of P.

In the classical statement of R.Isaacs, the aim of player P is to capture player E into a given circular neighborhood as soon as possible. The solution to the problem for some set of parameters was presented in the book of R.Isaacs. However, he was not sure about the structure of solutions in some rear region.



John Breakwell and his excellent pupils continued the investigations of R.Isaacs. Breakwell encouraged his pupils to explore more carefully the questionable region mentioned by Isaacs in his book. He proposed as well to study how the solution depends on the parameters of the problem.

Breakwell also stated the second problem where he reversed the goals of the players. Here player E tries to escape from the detection zone as soon as possible whereas player P hinders that. Thus the terminal set of this problem is not convex. It is the complement of the circle, the initial states are inside of the circle. Player E minimizes the time of attaining the terminal set, player P maximizes this time.

A.Merz. Solution of the first problem

The Homicidal Chauffeur – a differential game

A Dissertation
submitted to the department of aeronautics and astronautics
and the committee on graduate studies
of Stanford University
in partial fulfillment of the requirements
for the degree of
doctor of phylosophy

By
Antony Willitz Merz
March 1971

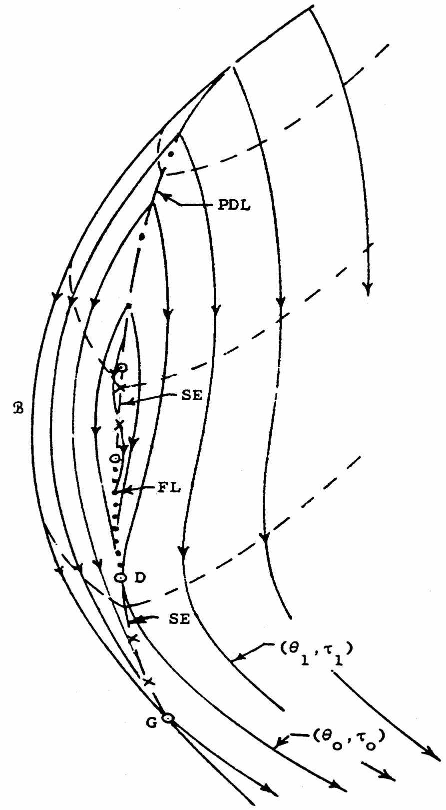
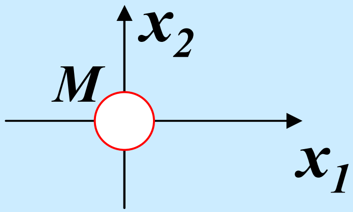


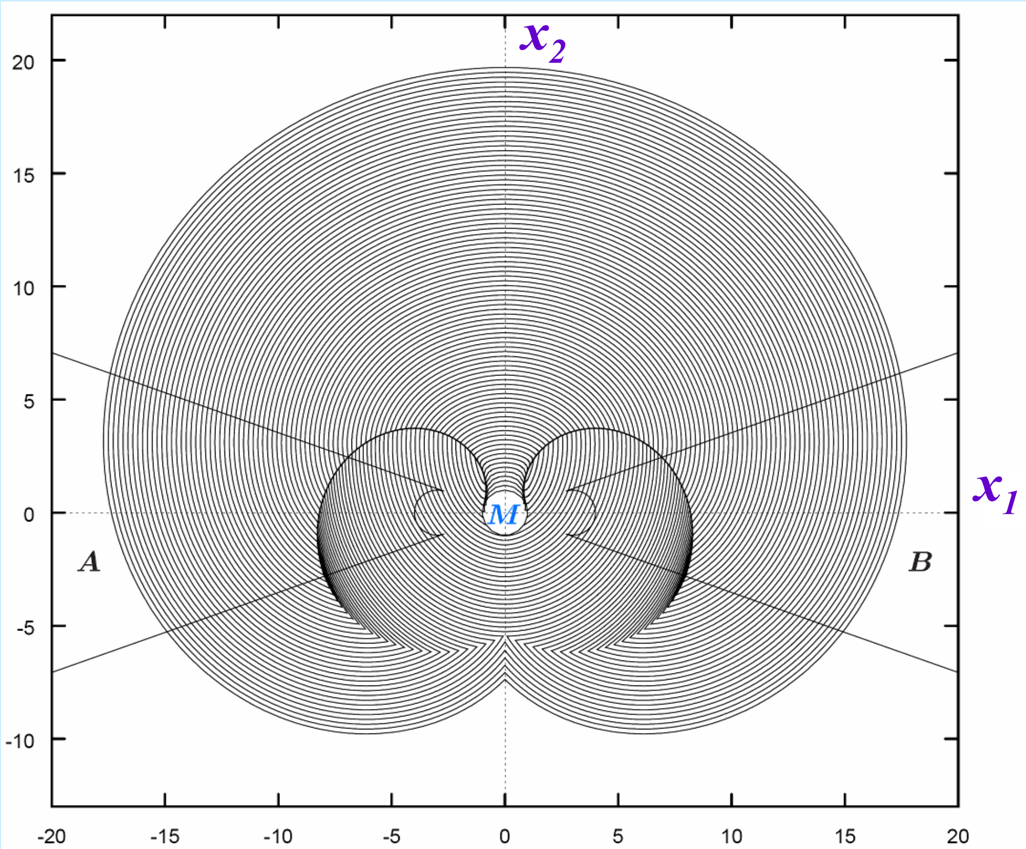
FIGURE 3.7. Trajectories Near the FL in Region II

Antony Merz has completely investigated the first problem in his dissertation. He classified all singularities depending on the parameters of the game. All types of singular lines – dispersal, equivocal, universal, and focal – are present in the homicidal chauffeur game. It is wondering how he was able to do this. Unfortunately, this work of Merz is not published up to now completely.

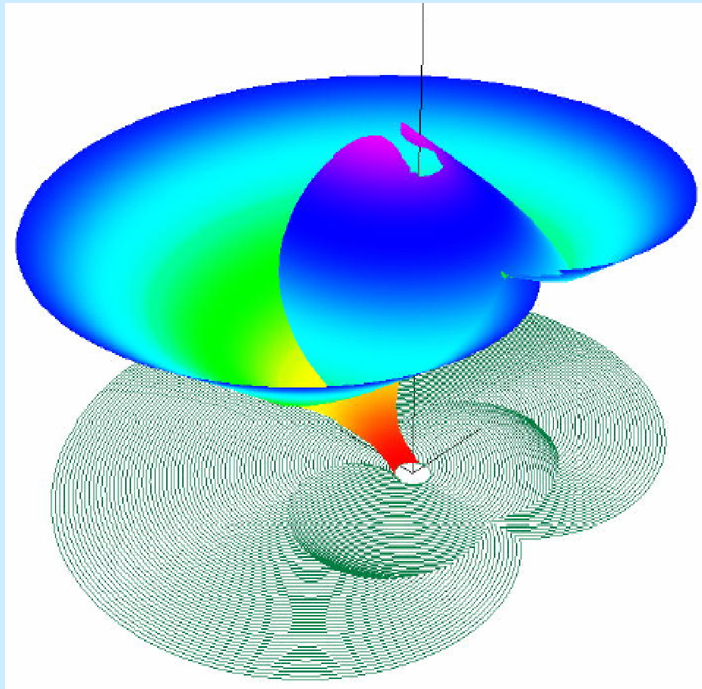
Terminal set is a circle with the center at the origin



Level sets

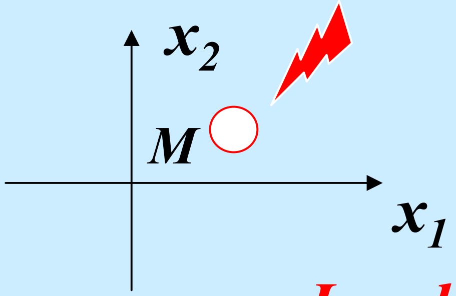


Graph of the value function



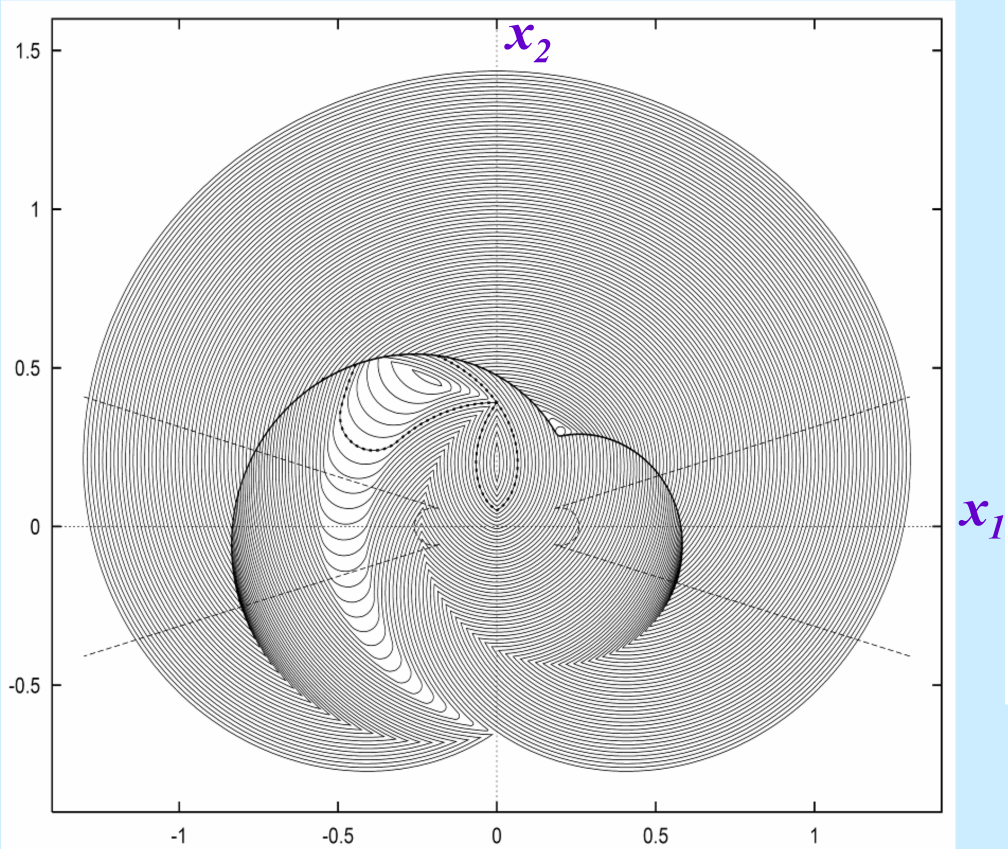
To the left, the level sets of the value function (isochrones) for a standard set of the problem parameters are shown. The terminal set is a circle centered at the origin. To the right, the graph of the value function is shown.

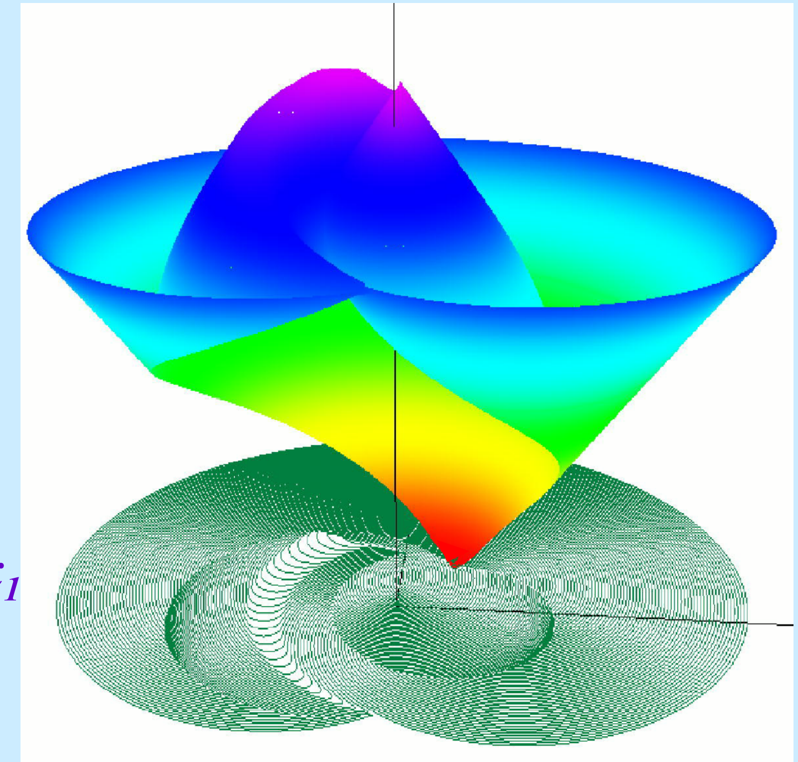
Shifted terminal set



Graph of the value function

Level sets





Shifting the terminal circle into the first quadrant violates the symmetry of the solution with respect to the axis x_2. Thereby, the structure of the level sets in the rear region below the terminal set becomes very interesting.

J.Lewin. Solution of the second problem

DECOY IN PURSUIT-EVASION GAMES

A DISSERTATION

submitted to the department of aeronautics and astronautics and the committee on graduate studies of Stanford University in partial fulfillment of the requirements for the degree of doctor of philosophy

by

Joseph Lewin

August 1973

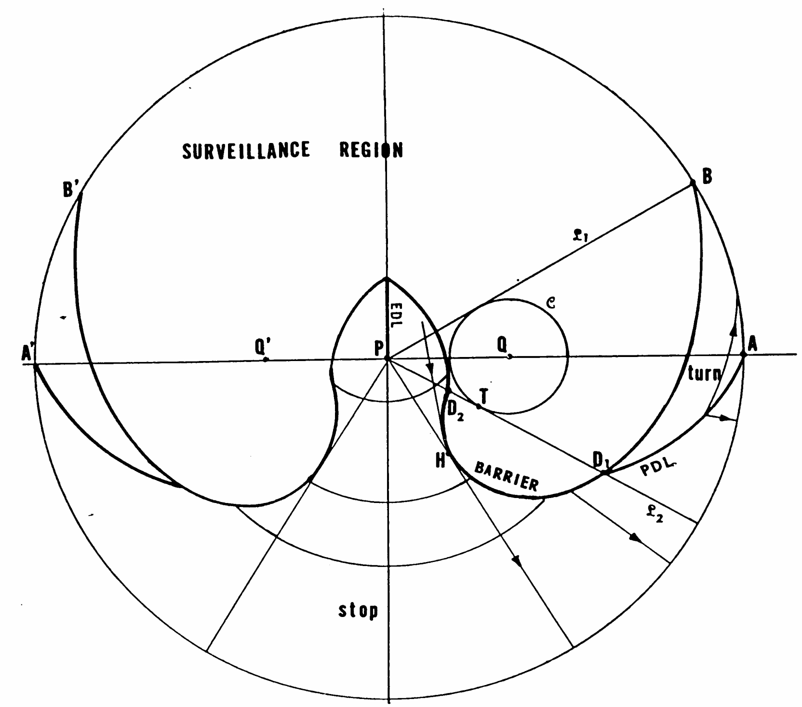
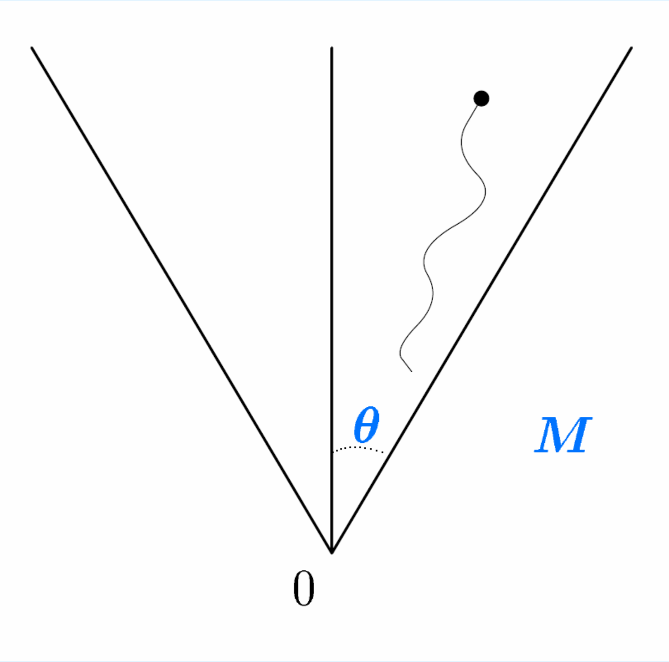


Fig. 5.21 $f_2(\gamma) < \beta \le f_3(\gamma)$

The solution to the second problem was given in the thesis of Joseph Lewin. He has also classified the development of singular lines depending on the parameters of the problem.

Conic surveillance-evasion game by J.Lewin and G.Olsder



E maximizes and P minimizes the time of attaining M

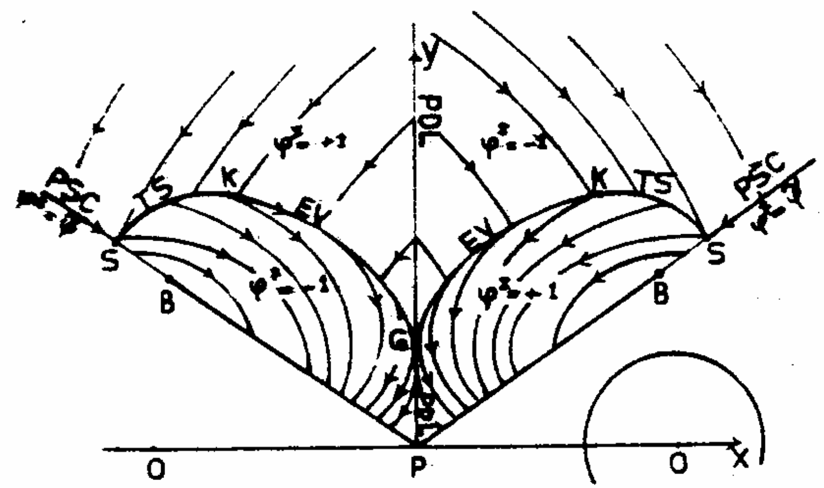
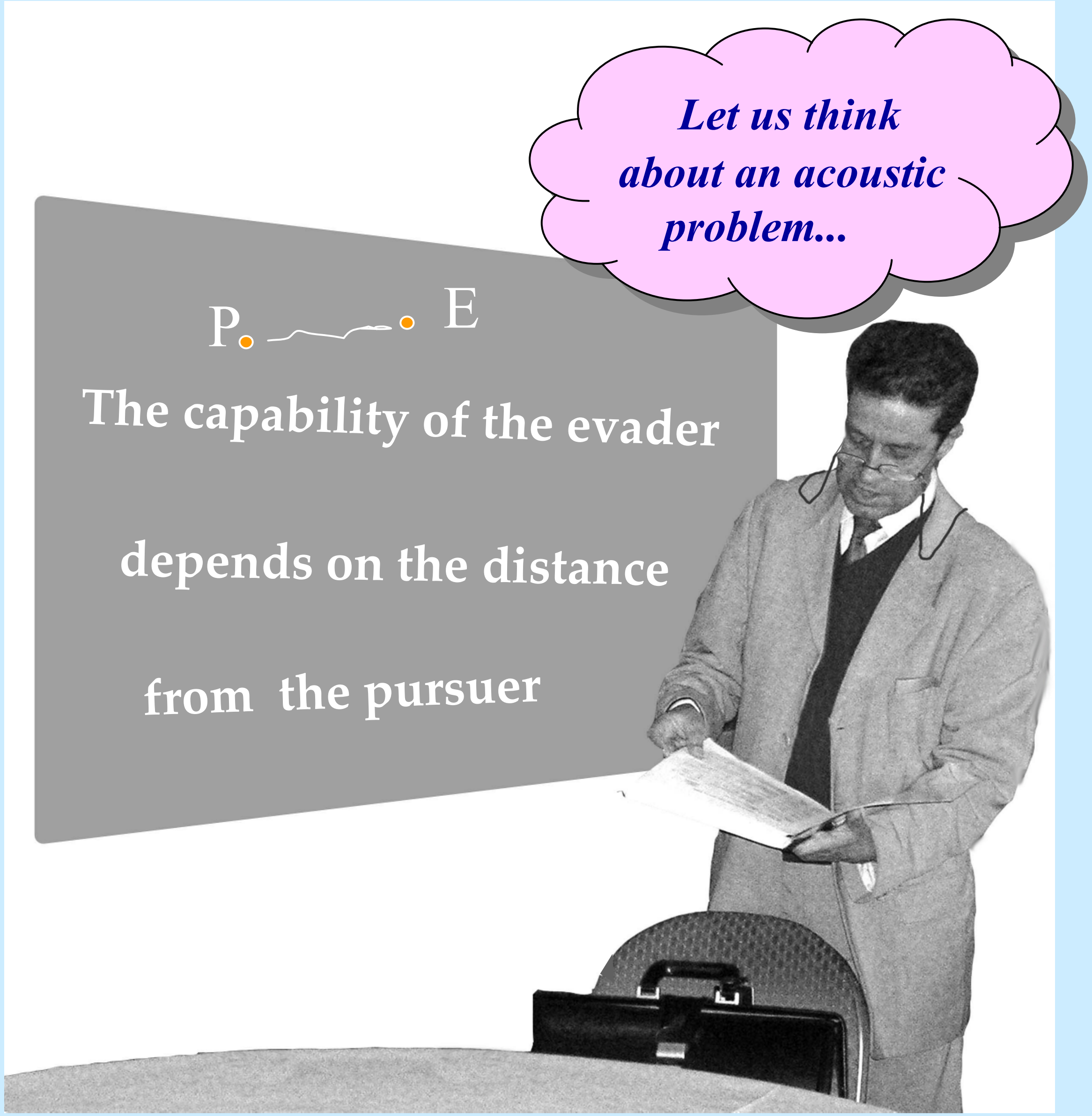


Fig. 8. $\pi/2 \geq \bar{\theta} \geq \pi/4$.

The surveillance-evasion game with the conic detection zone was studied in the paper by Joseph Lewin and Geert Jan Olsder published in JOTA.



Pierre Bernhard has proposed more complex dynamics for the homicidal chauffeur game: the constraint on the velocity of player E depends on the distance between P and E. For example, the evading submarine should reduce the noise if the pursuer is close to it. So one arrives at the acoustic homicidal chauffeur game.

Patrick Saint-Pierre, Marc Quincampoix, and Pierre Cardaliaguet started to think about the acoustic game. They were interested first of all in the structure of the capture set. Which parameters yield interesting structure of the capture set? For example, when holes in the capture set can appear?

They pointed out that the simplest way is to stretch the terminal set along x_1-axis, which was associated with a wide shoulders boxer.

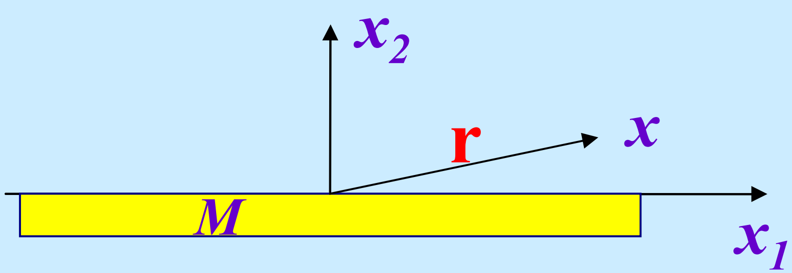
... in such a way a „boxer“ has appeared in the acoustic game with the dynamics of the homicidal chauffeur.

In this game, many interesting phenomena have been discovered – first of all the presence of a hole in the capture set.

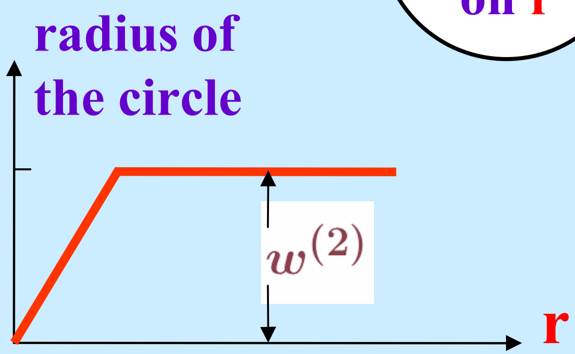
$$\dot{x}_1 = -w^{(1)} x_2 \, \varphi / R + v_1$$
$$\dot{x}_2 = \; w^{(1)} x_1 \, \varphi / R + v_2 - w^{(1)}$$

$$|\varphi| \leq 1 \, , \quad v \in$$

radius depends on r

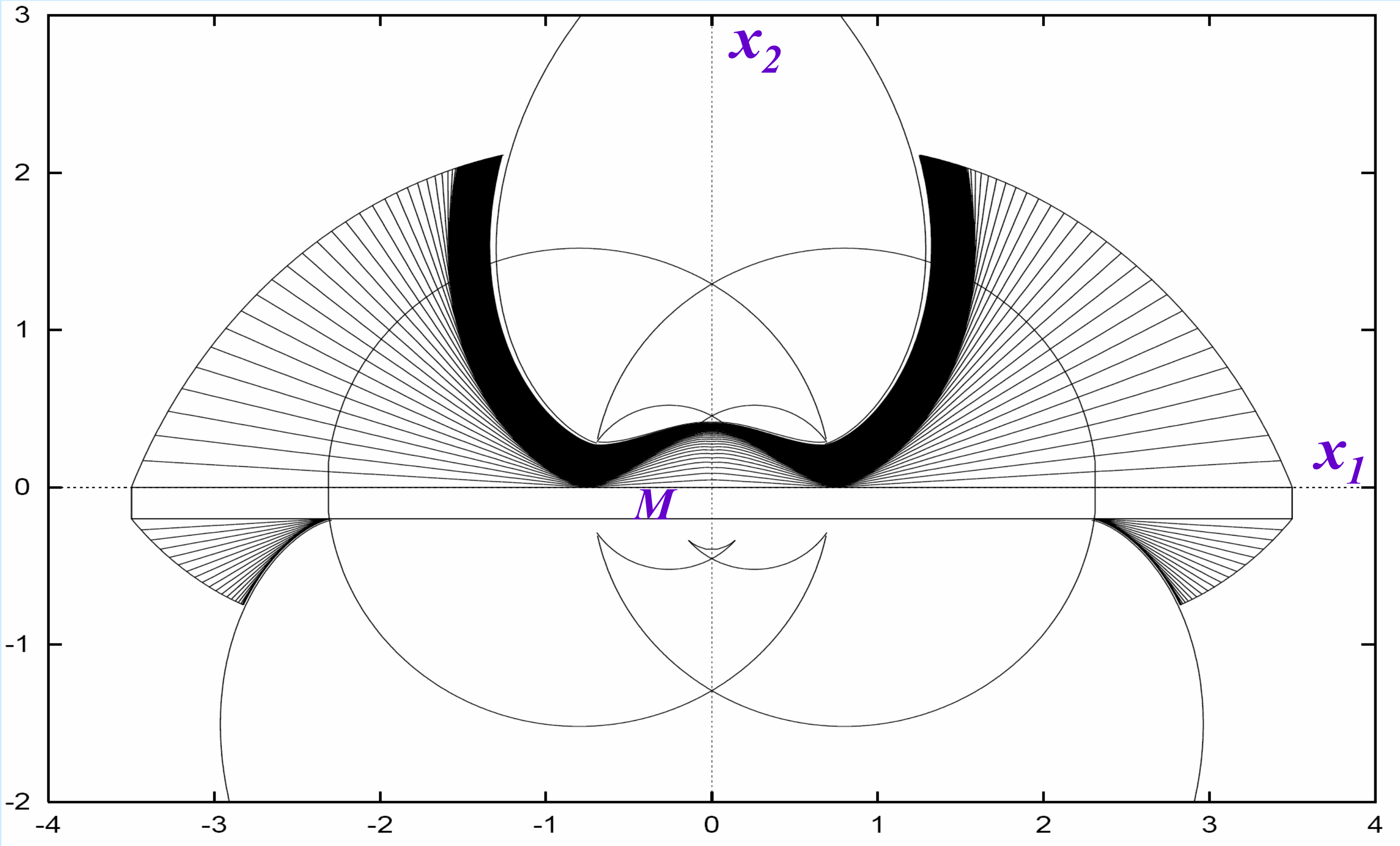


Terminal set



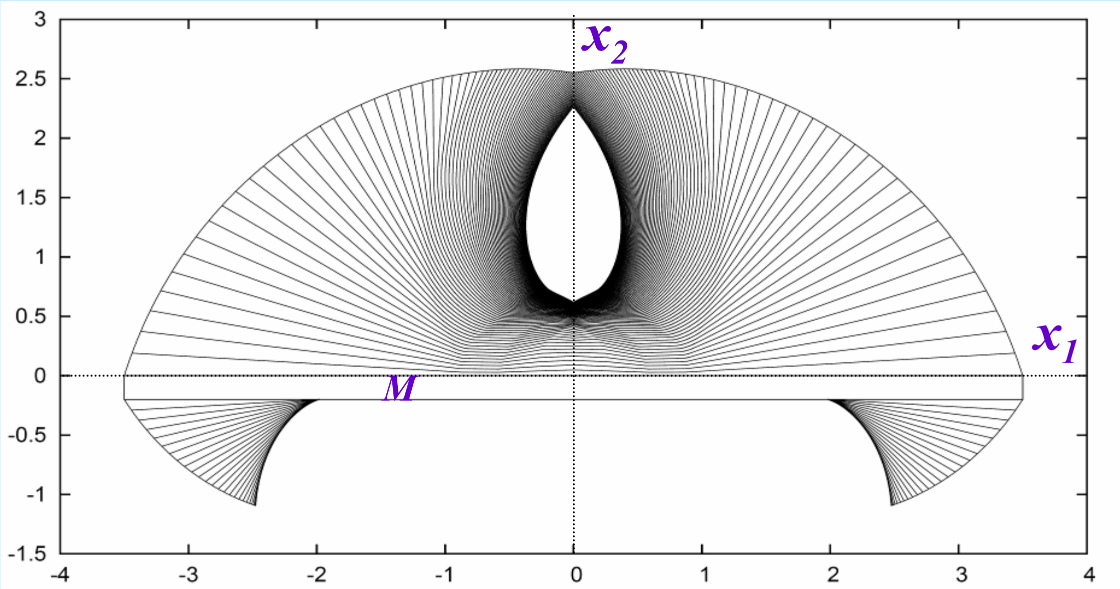
Thus, let the radius of the circular constraint of player E depends linearly from the distance (in reduced coordinates) to the origin if the distance is small. If the distance is greater than a given threshold, the radius of the circular constraint assumes its maximal constant value. The terminal set M is a rectangle stretched along x_1-axis.

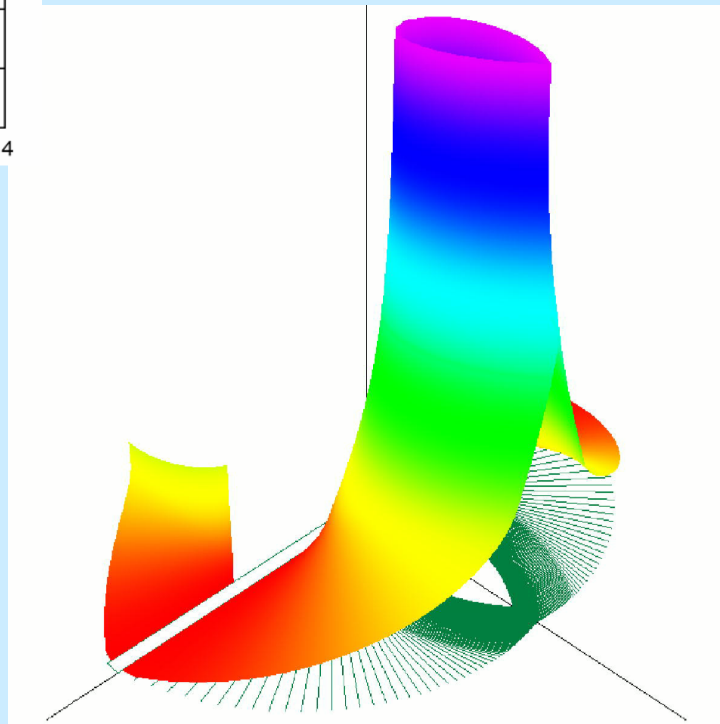
The hole arises in such a way



The maximal velocity of the evader is large.
No hole in the capture set arises

If the horizontal size of the terminal set M is fixed but the maximal radius $w^{(2)}$ of the circular constraint on the control of player E varies, then the level sets of the value function computed for a large enough value of the parameter $w^{(2)}$ look like the ones shown above. The dark region is a zone of the accumulation of fronts (isochrones). Here, the capture time tends to infinity. The capture set has no holes.





A hole where the value function V(x) is equal to infinity arises when decreasing the maximal velocity of the evader

Decreasing the capabilities of player E (i.e., decreasing $w^{(2)}$ with respect to the value utilized for the picture from page 15) gives rise to a hole in the capture set. It is important that the hole does not touch the terminal set. The opponent of the wide shoulders boxer can move around inside the hole unlimitedly long without being captured.

References

1. R.Isaacs (1951). Games of Pursuit. Scientific report of the RAND Corporation, Santa Monica.

2. R.Isaacs (1965). Differential Games. New York: John Wiley.

3. J.V.Breakwell and A.W.Merz (1969). Toward a complete solution of the homicidal chauffeur game, Proceedings of the 1st International Conference on the Theory and Applications of Differential Games, Amherst, Massachusetts.

4. A.W.Merz (1971). The Homicidal Chauffeur – a Differential Game. PhD Thesis, Stanford University.

5. J.Lewin (1973). Decoy in Pursuit-Evasion Games. PhD Thesis, Stanford University.

6. J.Lewin and J.V.Breakwell (1975). The surveillance-evasion game of degree, JOTA, Vol. 16, No. 3/4.

7. J.Lewin and G.J.Olsder (1979). Conic surveillance-evasion, JOTA, Vol. 27, No. 1.

8. P.Bernhard and B.Larrouturou (1989). Etude de la barriere pour un problema de fuite optimale dans le plan. Rapport de Recherche, INRIA, Sophia Antipolis.

9. P.Cardaliaguet, M.Quincampoix, and P. Saint-Pierre (1993). Some algorithms for differential games with two players and one target. Rapport de Recherche n°9343, CEREMADE, Paris.

10. P.Cardaliaguet, M.Quincampoix, and P. Saint-Pierre (1995). Numerical methods for optimal control and differential games. Ceremade CNRS URA 749, University of Paris-Dauphine.

The presentation is based on the references listed here.

Production

of

V.S.Patsko
Institute of Mathematics
and Mechanics
Russian Academy of Sciences
Ekaterinburg, Russia

V.L.Turova
Center of Advanced
European Studies and
Researh
Bonn, Germany

2005