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OPTIMAL STRATEGIES IN A PURSUIT PROBLEM WITH INCOMPLETE INFORMATION†

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A model pursuit problem with incomplete information is considered. The case when the information sets [1–5] are intervals is studied. The major part of the paper is devoted to proving the optimality of the proposed inverse connection strategies. Modelling results are presented.

One of the versions of the study of control problems with incomplete information on the phase vector is related to the change to the description of the state as a set of points in phase space compatible with the information on the behaviour of the system collected up to a given instant of time. Such sets are called information sets (IS). The problem can be stated as one of controlling the time variation of the IS with the view to minimizing some index. Since the IS depends on the current reading, minimax formulations arise naturally, in which the first player can exercise efficient control, while the other can choose the readings. Many papers (see, for example, [1–5]) have been devoted to the formalization of such problems. Optimal solutions for a number of problems of this type (in which the IS are intervals) were obtained in [3, 6, 7].

1. DETAILED DESCRIPTION OF THE PROBLEM

Two material points, the pursuer P and the evading party E, move in a plane. The pursuer measures the angular velocity of the line of sight at discrete instants of time t_i and, applying impulse control at these instants, attempts to reduce the magnitude of the miscalculation. Large miscalculation is regarded as unsatisfactory (high-accuracy guidance). The impulses act orthogonally to a direction prescribed in advance and constant in time. Let σ be the magnitude of a single impulse and N the total number of impulses. We denote by $k(t_i)$ the number of impulses of prescribed sign applied at a time t_i ($k(t_i)$ is either a negative integer or a positive integer or zero), subject to the constraint $\Sigma_i|k(t_i)| \leqslant N$. The evading party can alter its own velocity of motion by choosing the acceleration vector υ with components υ_x and υ_z, the control function υ being subject to the condition $|\upsilon| \leqslant \nu$.

The value of $\omega_m(t_i)$ measured at the time t_i and the true angular velocity $\omega(t_i)$ of the line of sight are related by $\omega_m(t_i) = \omega(t_i) + \xi(t_i)$. Here $\xi(t_i)$ is the reading error, about which it is known only that $|\xi(t_i)| \leqslant c$, where $c \geqslant 0$ is a given constant.

We shall assume that the nominal position $\tilde{E}$ of the evading party and its nominal velocity vector $\tilde{V}_E$ are specified at the initial time t_0. We identify the origin of the difference system of coordinates with the pursuer (Fig. 1). We take the x-axis to be in the opposite direction to the vector $\tilde{V} = \tilde{V}_E - V_P(t_0)$, where $V_P(t_0)$ is the initial velocity vector of the pursuer. Suppose that the pursuer chooses $V_P(t_0)$ at t_0 so that the x-axis passes through $\tilde{E}$. We denote by z the axis perpendicular to x. The directions of the axes are constant with respect to t. We assume that the impulses act along the z-axis (economical use of resources in the course of conducting the pursuit). The initial position $E(t_0)$ of the evading party and the initial velocity vector $V_E(t_0)$ can differ from the preliminary prescribed values $\tilde{E}$ and $\tilde{V}_E$. We assume, however, that these differences are not too large. Let $\alpha(t)$ be the current sighting angle, let $V_x(t)$ be the component of the velocity difference along the x-axis, and let e be the length of $\tilde{V}$.

In the x, z-coordinates the dynamic equations have the form

$$\ddot{z}(t) = \upsilon_z - \sigma\sum_i k(t_i)\delta(t - t_i), \qquad \ddot{x}(t) = \upsilon_x \tag{1.1}$$

$$\sum_i |k(t_i)| \leqslant N, \qquad |\upsilon| \leqslant \nu$$

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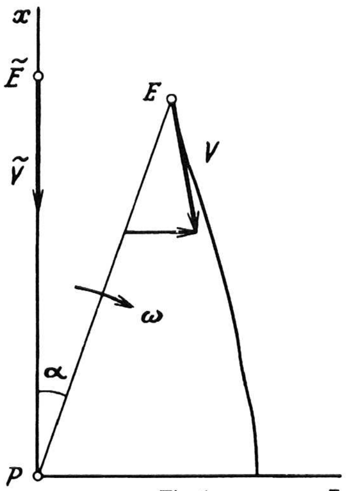


Fig. 1.

The impulse interaction at time t_i alters the velocity $\dot{z}$ by a jump of magnitude $-\sigma k(t_i)$.

2. THE DYNAMICS OF AN AUXILIARY DIFFERENTIAL GAME

Using the relationship $z(t) = x(t)\text{tg}\,\alpha(t)$, we rewrite the original system (1.1) in α, ω, x, V_x coordinates. We get

$$\begin{aligned}
&\dot{\alpha}(t) = \omega(t) \\
&\dot{\omega}(t) = -2V_x(t)\omega(t)/x(t) - 2\,\text{tg}\,\alpha(t)\omega^2(t) - \sin\alpha(t)\cos\alpha(t)\upsilon_x / x(t) + \\
&+\cos^2\alpha(t)\upsilon_z / x(t) - (\sigma\cos^2\alpha(t)/x(t))\sum_i k(t_i)\delta(t-t_i) \qquad (2.1) \\
&\dot{x}(t) = V_x(t), \qquad \dot{V}_x(t) = \upsilon_x
\end{aligned}$$

The system in terms of the equivalent coordinates proves convenient because the observed value ω appears among the phase variables. Moreover, transition to various auxiliary problems differing by the degree of simplification becomes possible. In the present paper we shall confine ourselves to the simplest problem. We then use the resulting control algorithm in the original problem. A more complex auxiliary problem is considered in [8]. However, no optimal strategies have been found in the more complex case.

We assume that the variation of the velocity difference vector arising during the motion because of the control exercised by the pursuer and the evading party is relative small (weak control). Because miscalculations which are not too large are of interest here, this assumption implies that α must be small during a relatively long interval of motion starting from the initial time. Indeed, suppose that α is large at some time t_*. Since the cone on which the future motion can evolve is small and the axis of the cone is close to being vertical, the miscalculation at the end will clearly be significant; the longer the time between t_* and the end the larger the miscalculation. We therefore assume α to be small. In the simplified formulation of the auxiliary problem we substitute zero for sin α and tan α, and one for cos α. Since ω is the observed value and α does not appear in the relationship for $\dot{\omega}$ any more, the equation $\dot{\alpha}(t) = \omega(t)$ can be discarded. We can therefore reduce the dimension of the phase vector by one.

Let us make additional simplifications. As a result of the weak control assumption, the miscalculation for a specific motion can be approximated (taking into account that $\tilde{V}$ is directed along the x-axis) by computing the modulus of the z coordinate at the time when $x(t) = 0$. It turns out that the variation of the velocity along the z-axis has a more significant effect on the magnitude of the miscalculation than that along the x-axis. It follows that, simplifying (2.1), we can assume $V_x(t_0)$ to be known precisely and equal to $\tilde{V}_x = -e$, and the control function υ_x can be assumed to be identically zero. The variation of the ω, x coordinates can be described by

$$\begin{aligned}
&\dot{\omega}(t) = 2e\omega(t)/x(t) + \upsilon_z / x(t) - (\sigma / x(t))\sum_i k(t_i)\delta(t-t_i) \\
&x(t) = x(t_0) - e(t-t_0)
\end{aligned}$$

Finally, we assume that $x(t_0)$ is specified precisely in the auxiliary problem. The behaviour of the system in the x direction is then uniquely defined. Denoting the final instant by ϑ, we rewrite the relationship for $\dot\omega(t)$ in the form

$$\dot\omega(t) = 2\omega(t)/(\vartheta - t) + \upsilon_z / e(\vartheta - t) - (\sigma / e(\vartheta - t))\sum_i k(t_i)\delta(t - t_i) \tag{2.2}$$

The impulse interaction at t_i causes ω to jump by $-\sigma k(t_i)/e(\vartheta - t_i)$. Moreover, $\Sigma_i |k(t_i)| \leqslant N$.

We shall adopt (2.2) as a description of the dynamics of the auxiliary differential game. Here is a list of indeterminacies

$$\omega_m(t_i) = \omega(t_i) + \xi(t_i), \quad |\xi(t_i)| \leqslant c, \quad |\upsilon_z| \leqslant \nu, \quad \omega(t_0) \in A \tag{2.3}$$

where A is an interval containing all values of the angular velocity at t_0 possible because of the initial indeterminacy of $x, z, \dot x, \dot z$ in the difference system (1.1). The variation ω is described by

$$\omega(t) = \omega(\hat t)\frac{(\vartheta - \hat t)^2}{(\vartheta - t)^2} + \frac{1}{e(\vartheta - t)^2}\int_{\hat t}^{t} (\vartheta - \tau)\upsilon_z(\tau)d\tau - \frac{\sigma}{e(\vartheta - t)^2}\sum_{t_i \in [\hat t, t]} k(t_i)(\vartheta - t_i)$$

The maximum (minimum) value of the second component is attained for $\upsilon_z \equiv \nu$ ($\upsilon_z \equiv -\nu$). We set

$$\zeta(\hat t, t) = \frac{1}{e(\vartheta - t)^2}\int_{\hat t}^{t} (\vartheta - \tau)\nu d\tau = \frac{\nu(t - \hat t)}{e(\vartheta - t)^2}\left(\vartheta - \frac{1}{2}(\hat t + t)\right)$$

3. FORMALIZATION OF A DIFFERENTIAL GAME WITH INCOMPLETE INFORMATION

We will now formulate an auxiliary differential game of two players, in which the state at time t_i is represented by a pair consisting of the information set of the ω-axis and the remaining number of impulses.

As the initial information set $I_-(t_0)$ we take an arbitrary interval on the ω-axis. Consider the time t_i and the interval $I_-(t_i)$, $i \geqslant 0$. Let $H(t_i) = \{\omega: |\omega - \omega_m(t_i)| \leqslant c\}$ be the indeterminacy set corresponding to the reading $\omega_m(t_i)$. We set $I(t_i) = I_-(t_i) \cap H(t_i)$ and assume that this product is non-empty. At the time t_i, after taking a reading, the first player, who chooses the impulses, can exercise his control. A rigid translation by $-\sigma k(t_i)/e(\vartheta - t)$ will then carry the interval $I(t_i)$ into $I_+(t_i)$.

We define $I_-(t_{i+1})$ to be the prediction of the position of (2.2) at t_{i+1} given that $I_+(t_i)$ is the state at time t_i and no control is exercised by the first player during $(t_i, t_{i+1}]$. Let ω_* and ω^* be the left and right ends of the intervals under consideration. The coordinate $\omega_{*-}(t_{i+1})$ of the left end of $I_-(t_{i+1})$ is equal to

$$\omega_{*-}(t_{i+1}) = \omega_{*+}(t_i)\frac{(\vartheta - t_i)^2}{(\vartheta - t_{i+1})^2} - \zeta(t_i, t_{i+1}) \tag{3.1}$$

The coordinate $\omega^*_-(t_{i+1})$ of the right end is defined by (3.1) with $\omega_{*+}(t_i)$ replaced by $\omega^*_+(t_i)$ and with the sign of the last term altered.

Each of the sets $I_-(t_i)$, $I(t_i)$, $I_+(t_i)$ will be called an information set (prior to the reading, after the reading, and after the impulses). We also call $I_-(t_i)$ the prediction set. A recurrent sequence of information sets is thus defined. By the motion in the auxiliary game we mean the time variation of the information set and the remaining impulse number. The first player can choose the controlling impulses, while the other one can control the readings. We will take into account the effect of υ_z when constructing the prediction set. When $c = 0$ the game under consideration turns into one with complete information. Then the measurement at time t_i entails the choice of a point in $I_-(t_i)$, which corresponds to the action of some control function $\upsilon_z(t)$ in the interval $[t_{i-1}, t_i)$.

We denote by ω_c the centre of the information interval, and by b its half-width. We have $I = (\omega_c, b)$ and $I_\pm = (\omega_{c\pm}, b_\pm)$. Let $n_+(t_i) = n(t_i) - k(t_i)$. If $k(t_i) = 0$, then $I_+(t_i) = I(t_i)$ and $n_+(t_i) = n(t_i)$.

We take (t_i, n, I) to be the position of the game for the first player, t_i being the time, n the remaining impulse number, and I the information set after the reading. A rule U: $(t_i, n, I) \to k$ which assigns the number of impulses of given sign such that $|k| \leqslant n$ to a position of the game will be called an admissible strategy of the first player. We call (t_i, n, I_-) the position of the game for the second player. Here I_- is the information set before the reading (the prediction set). A rule Ω: $(t_i, n, I_-) \to \omega_m$ assigning a reading ω_m to the position will be called an admissible strategy of the second player. We require that $I_- \cap H \neq \varnothing$, where H is the indeterminacy set constructed from ω_m.

Specifying a pair U, Ω of admissible strategies, a step Δ of the discrete observation and control scheme, as well as the initial position $(t_0, n(t_0), I_-(t_0))$, we can talk of the motion of the system in time. Let us define the cost functional. For arbitrary ω, t $(t < \vartheta)$ we set $\Pi(\omega, t) = |\omega| e(\vartheta - t)^2$, which is an approximation of the passive predicted miscalculation at ω, t, i.e. the distance of the free motion of system (2.2) from zero at the time when $x = 0$ (for an exact computation, one would need to prescribe, in addition to ω, t, an angle α: $|\omega| e(\vartheta - t)^2/(\cos^2 \alpha)$. Let $\hat{\Pi}(\omega, t) = \Pi(\omega, t) + \nu(\vartheta - t)^2/2$ be the maximum predicted miscalculation.

We fix $\varepsilon \geqslant 0$. We denote by t_ε the closest time t_i to the left of $\vartheta - \varepsilon$. The number

$$\Phi(t_0, n(t_0),\ I_-(t_0),\ U, \Omega, \Delta, \varepsilon) = \max\{\hat{\Pi}(\omega, t_\varepsilon):\ \omega \in I_+(t_\varepsilon)\}$$

will be called the miscalculating corresponding to the initial position $(t_0, n(t_0), I_-(t_0))$ and to U, Ω, Δ, and ε.

The best guarantees for the players are defined by

$$\Gamma^{(1)}(t_0, n(t_0),\ I_-(t_0)) = \inf_U \overline{\lim_{\varepsilon \to 0}} \cdot \overline{\lim_{\Delta \to 0}} \sup_\Omega \Phi(t_0, n(t_0),\ I_-(t_0),\ U, \Omega, \Delta, \varepsilon)$$

$$\Gamma^{(2)}(t_0, n(t_0),\ I_-(t_0)) = \sup_\Omega \underline{\lim_{\varepsilon \to 0}}\ \underline{\lim_{\Delta \to 0}} \inf_U \Phi(t_0, n(t_0),\ I_-(t_0),\ U, \Omega, \Delta, \varepsilon)$$

Let us introduce the strategy U_0 of the first player. We set

$$k^0 = \left[\frac{\omega_c(t_i) e(\vartheta - t_i)}{\sigma} \right] \tag{3.2}$$

Here the square brackets denote the integer part. We specify U_0 as a function assigning to (t_i, n, I) the number k^0 computed from (3.2) if $|k^0| \leqslant n$ and n sign k^0 if $|k^0| > n$. Since $\sigma/e(\vartheta - t)$ is the variation of (the modulus of) ω due to a single impulse, the computations using (3.2) can be regarded as equating ω_c (taking the sign into account) with the threshold $\sigma/e(\vartheta - t)$ equal to the effectiveness of a single impulse. Figure 2 illustrates this: it presents a possible variation of the information interval I under the action of U_0. The directions of impulses are represented by horizontal arrows.

We shall introduce the strategy Ω^0 of the second player. Let ω^0 be the end-point of $I_-(t_i)$ at which the maximum of $\hat{\Pi}(\omega, t_i)$ for $\omega \in I_-(t_i)$ is attained. The point ω^0 coincides with the point of $I_-(t_i)$ furthest from zero. We define Ω^0 as follows: if ω^0 is the right end of $I_-(t_i)$, then we set $\omega_m = \omega^0 - c$; if ω^0 is the left end, then $\omega_m = \omega^{(1)} = c$. This means that the reading is taken in such a way as to ensure that the "worst" point ω^0 from the prediction interval $I_-(t_i)$ falls into $I(t_i)$ and to obtain the maximum possible length of $I_-(t_i)$.The strategy Ω^0 is independent of $n(t_i)$ at the time t_i.

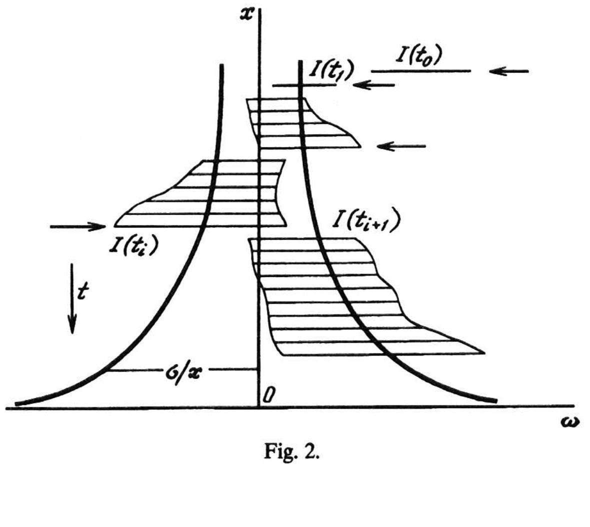


Fig. 2.

4. THE OPTIMALITY OF U_0 AND Ω_0

The problem being studied is symmetric with respect to ω. We therefore assume that the right end of the information interval lies no closer to zero than the left one. If necessary, we replace the interval by the symmetric one.

4.1. *Standard motion*

We shall determine the motion $t \to (n(t), I(t))$, which will be called the standard motion (StM). It represents the variation of the information interval and the remaining impulse number under continuous observation (where Δ is negligibly small) and control according to the strategies U^0 and Ω^0. We assume that the right end of the current indeterminacy set coincides with the right end of the information interval. The component $n(\cdot)$ characterizes the impulse outlay. In the description of the StM we shall use the previous notation: ω_* and ω^* for the left and right ends, etc.

Assume that $\omega^*(\hat{t}) \geqslant 0$ at some initial time $\hat{t}$ and $\omega^*(\hat{t}) \geqslant \omega^*(\hat{t}) - 2c$. Let $t \to \omega^*(t)$ be a piecewise continuous function describing the variation of the right end of I and let $t \to \omega_*(t)$ be the analogous function for the left end. Next, let the following conditions (1)–(3) be satisfied on the continuous pieces of $\omega^*(\cdot)$:

$$1.\ \dot{\omega}^*(t) = 2\omega^*(t)/(\vartheta - t) + \nu/e(\vartheta - t),$$

$$2.\ \omega_*(t) \geqslant \omega^*(t) - 2c,$$

$$3.\ \text{if}\ \ \omega_*(t) > \omega^*(t) - 2c, \quad \text{then} \quad \dot{\omega}_*(t) = 2\omega_*(t)/(\vartheta - t) - \nu/e(\vartheta - t).$$

We adopt the following conditions (4)–(6) for the discontinuities of $\omega^*(\cdot)$.

4. If $t \geqslant \hat{t}$ is the instant of discontinuity for $\omega^*(\cdot)$, then the jump $\Delta\omega^*(t) = \omega^*_+(t) - \omega^*(t)$ is equal to $-\sigma k(t)/e(\vartheta - t)$ for some positive integer $k(t) \leqslant n(t)$, i.e. the discontinuity of $\omega^*(\cdot)$ corresponds to $k(t)$ impulses to the left. The new value (the right-hand limit with respect to t) for the left edge is computed from the formula $\omega_{*+}(t) = \omega_*(t) + \Delta\omega^*(t)$.

5. The discontinuities of $\omega^*(\cdot)$ for $t > \hat{t}$ occur at times when the centre of the interval coincides with the threshold $\sigma/e(\vartheta - t)$. The jump corresponds to a single impulse to the left.

6. When a jump occurs at $\hat{t}$, then $k(\hat{t})$ can be defined as the smallest positive integer $k \leqslant n(\hat{t})$ such that $\omega_{c+}(\hat{t}) = \omega_c(\hat{t}) - \sigma k/e(\vartheta - \hat{t}) < \sigma/e(\vartheta - \hat{t})$. In other words, $k(\hat{t})$ is the smallest k for which the new centre $\omega_{c+}(\hat{t})$ falls below the threshold $\sigma/e(\vartheta - \hat{t})$. If k exceeds $n(\hat{t})$, then we set $k(\hat{t}) = n(\hat{t})$.

Let us specify $n(\cdot)$:

7. $t \to n(t)$ is a piecewise constant function defined by $n(\hat{t}) = n(\hat{t}) - \Sigma_{tj<t}k(t_j)$, where $t_j \geqslant \hat{t}$ are the instants of discontinuity of $\omega^*(\cdot)$.

The pair $(n(\cdot), I(\cdot))$ satisfying conditions (1)–(7) will be called the StM originating at $(\hat{t}, n(\hat{t}), I(\hat{t}))$. With the StM we associate the value $F(\hat{t}, n, I)$ of a hypothetical miscalculation function, which we define as the maximum predicted miscalculation computed for the StM at the time t_j of the last impulse: $F(\hat{t}, n, I) = \hat{\Pi}(\omega^*_+(t_f), t_f)$.

4.2. *Auxiliary assertions*

We shall state some properties of the motions of (2.2), the StM, and the hypothetical miscalculation functions.†

1. Let $\omega^{(1)}(\cdot)$, $\omega^{(2)}(\cdot)$ be the solutions of the differential equation (2.2) in $[\hat{t}, t']$ originating from the state $\omega^{(1)}(\hat{t}) = \omega^{(2)}(\hat{t})$, due to the control interactions $\upsilon_z^{(1)}(\cdot), k^{(1)}(\cdot)$ and $\upsilon_z^{(1)}(\cdot) = \upsilon_z^{(2)}(\cdot), k^{(2)}(\cdot)$. We make the following assumptions: $\upsilon_z^{(1)}(\cdot) = \upsilon_z^{(2)}(\cdot)$; $k^{(1)}(t)$; takes positive integral values in a discrete set of times $t_1, \ldots, t_d$ and vanishes at all other times; $k^{(2)}(t) = 0$ for $t \in [\hat{t}, t')$ and $k^{(2)}(t') = \Sigma_{s=1}^{d} k^{(1)}(t_s)$. Then $\omega^{(1)}(t') \leqslant \omega^{(2)}(t')$. This becomes an equality only when $d = 1$, $t_d = t'$.

2. Suppose that the inequality $\omega_*(t') > \omega_*(t') - 2c$ is satisfied for the StM at some instant $t' > t_0$. Then $\omega_*(t) > \omega^*(t) - 2c$ for $t \in [t_0, t')$. Up to the time t' the half-width $b(t)$ evolves inside any interval without impulses at the maximum possible rate described by $\dot{b}(t) = 2b(t)/(\vartheta - t) + \nu/e(\vartheta - t)$.

†For a detailed presentation see: KUMKOV S. I. and PATSKO V. A., A model pursuit problem with incomplete information. Preprint, Inst. Matem. i Mekh. Ural'sk. Otd. Ross. Akad. Nauk, Ekaterinburg, 1993.

3. Suppose that there are no impulses in $(\hat{t}, t')$ for a motion according to a discrete scheme given an arbitrary strategy U of the first player and the strategy Ω^0 of the second player. We assume that $\omega_*(t') > \omega^*(t') - 2c$. Then $\omega_*(t) > \omega^*(t) - 2c$ for $t \in (\hat{t}, t')$ and $b(t)$ evolves in $(\hat{t}, t')$ at the maximum possible rate given by $\dot{b}(t) = 2b(t)/(\vartheta - t) + \nu/e(\vartheta - t)$.

4. The hypothetical miscalculation function F is constant along the StM.

We assert the monotonicity of the hypothetical miscalculation function in the following lemma. We set $F^{(i)} = F(t, n^{(i)}(t), I^{(i)}(t)), i = 1, 2$.

Lemma 1. Let $(t, n^{(1)}(t), I^{(1)}(t))$ and $(t, n^{(2)}(t), I^{(2)}(t))$ be two positions at time t such that $n^{(1)}(t) = n^{(2)}(t)$, $\omega_c^{(1)}(t) \geqslant \omega_c^{(2)}(t)$, $b^{(1)}(t) \geqslant b^{(2)}(t)$. Then

$$F^{(2)} \leqslant F^{(1)} \leqslant F^{(2)} + (\omega^{(1)*}(t) - \omega^{(2)*}(t))e(\vartheta - t)^2 \tag{4.1}$$

The inequality $F^{(2)} \leqslant F^{(1)}$ is also satisfied if $n^{(1)}(t) \leqslant n^{(2)}(t)$ and $I^{(1)}(t) \supset I(2)(t)$.

The following two assertions are concerned with the values $F(t, n(t), I(t))$, $F(t, n(t) - 1, I_+(t))$ of F before and after a single impulse.

5. Let $(t, n(t), I(t))$ be an arbitrary position and let a single impulse to the right be applied. Then F does not decrease after the impulse.

6. Let $(t, n(t), I(t))$ be a position such that $\omega_c(t) < \sigma/e(\vartheta - t)$ (the centre below the threshold) and let a single impulse to the left be applied. Then F does not decrease after the impulse.

4.3. *Saddle point*

Let $K = [\omega_*, \omega^*]$ be an arbitrary interval. We denote by $Y(K)$ the interval whose right end coincides with ω^* and the left end is defined to be $\max\{\omega_*, \omega^* - 2c\}$.

Lemma 2. The inequality

$$\Phi(t_0, n(t_0), I_-(t_0),\ U, \Omega^0, \Delta, \varepsilon) \geqslant F(t_0, n(t_0), Y(I_-(t_0))) \tag{4.2}$$

is satisfied for any initial position $(t_0, n(t_0), I_-(t_0))$, any strategy U of the first player, and any Δ and ε.

Proof. By the real motion (RM) we mean the motion according to the discrete scheme and strategies U and Ω^0. The symbols referring to the RM will be denoted by a bar. Those referring to the auxiliary StM will be denoted by a tilde. The main idea is that F is non-decreasing along the RM. We specify the times $\tau_1, \ldots, \tau_m$ when the impulses act on the RM.

A. We shall study the variation of F along the RM in an interval from τ_s to τ_{s+1}. Under the action of the impulses $\bar{I}(\tau_s)$ becomes $\bar{I}_+(\tau_s)$. Properties (4)–(6) imply that

$$F(\tau_s, \bar{n}_+(\tau_s),\ \bar{I}_+(\tau_s)) \geqslant F(\tau_s, \bar{n}(\tau_s),\ \bar{I}(\tau_s)), \quad s = 1, \ldots, m \tag{4.3}$$

We shall establish the inequality

$$F(\tau_{s+1}, \bar{n}(\tau_{s+1}),\ \bar{I}(\tau_{s+1})) \geqslant F(\tau_s, \bar{n}_+(\tau_s),\ \bar{I}_+(\tau_s)), \quad s = 1, \ldots, m-1 \tag{4.4}$$

We start at StM from $(\tau_s, \bar{n}_+(\tau_s), \bar{I}_+(\tau_s))$ and continue up to the time τ_{s+1}. Let $g_s \geqslant 0$ be the number of impulses of the StM in $[\tau_s, \tau_{s+1})$. With the position $(\tau_{s+1}, \tilde{n}(\tau_{s+1}), \tilde{I}(\tau_{s+1}))$ we associate $(\tau_{s+1}, \tilde{n}^1(\tau_{s+1}), \tilde{I}^1(\tau_{s+1}))$, where $\tilde{n}^1(\tau_{s+1}) = \tilde{n}^1(\tau_{s+1}) + g_s = \bar{n}_+(\tau_s)$. The right end $\tilde{\omega}^{1*}(\tau_{s+1})$ of the interval $\tilde{I}^1(\tau_{s+1})$ is obtained by moving the impulses at $\tilde{\omega}^*(\tau_{s+1})$ to the right by g_s, and the half-width $\tilde{b}^1(\tau_{s+1})$ is equal to $\tilde{b}(\tau_{s+1})$. We have $F(\tau_{s+1}), \tilde{n}^1(\tau_{s+1}), \tilde{I}^1(\tau_{s+1})) = F(\tau_{s+1}, \tilde{n}^1(\tau_{s+1}), \tilde{I}^1(\tau_{s+1}))$ (briefly, $\tilde{F}^1 = \tilde{F}$).

We shall prove the inequality $F(\tau_{s+1}), \bar{n}(\tau_{s+1}), \bar{I}^1(\tau_{s+1})) \geqslant F(\tau_{s+1}), \tilde{n}^1(\tau_{s+1}), \tilde{I}^1(\tau_{s+1}))$ ($\bar{F} \geqslant \tilde{F}^1$). The argument τ_{s+1} will be omitted for brevity. Since the second player uses the strategy Q^0, the right end of the information interval for the RM moves to the right at the maximum possible speed. By (1), we get $\tilde{\omega}^{1*} \leqslant \bar{\omega}^*$. If $\tilde{\omega}_c^1 \geqslant \bar{\omega}_c$, then $\tilde{I}^1 \subset \bar{I}$ and the required inequality $\bar{F} \geqslant \tilde{F}^1$ follows from the second part of Lemma 1. Let $\tilde{\omega}_c^1 < \bar{\omega}_c$. When $\tilde{b}^1 \leqslant \bar{b}$, we invoke the left inequality in (4.1). We assume that $\bar{b} < \tilde{b}^1$. Then $\bar{b} < c$. On the basis of (3), we find that the half-width $\bar{b}(t)$ evolves at the maximum possible speed in (τ_s, τ_{s+1}). Therefore $\bar{b} \geqslant \tilde{b} = \tilde{b}^1$. This is a contradiction.

It follows that $\bar{F} \geqslant \tilde{F}^1$. Since $\tilde{F}^1 = \tilde{F}$, we have $\bar{F} \geqslant \tilde{F}$. By the equality $F(\tau_{s+1}, \tilde{n}(\tau_{s+1}), \tilde{I}(\tau_{s+1})) = F(\tau_s, \bar{n}_+(\tau_s), \bar{I}_+(\tau_s))$ the latter implies (4.4).

By analogy with (4.4), one can prove the following inequality in the case when $\tau_1 > t_0$ (i.e. when there is no impulse at the initial instant t_0)

$$F(\tau_1,\bar{n}(\tau_1),\bar{I}(\tau_1)) \geq F(t_0,\bar{n}(t_0),\bar{I}(t_0)) \tag{4.5}$$

Here the StM originates from $(t_0, \bar{n}(t_0), \tilde{I}(t_0))$.

B. From (4.3)–(4.5), we get

$$F(\tau_m,\bar{n}_+(\tau_m),\ \bar{I}_+(\tau_m)) \geq F(\tau_m,\bar{n}(\tau_m),\ \bar{I}(\tau_m)) \geq \ldots \geq$$
$$\geq F(\tau_{s+1},\bar{n}(\tau_{s+1}),\ \bar{I}(\tau_{s+1})) \geq F(\tau_s,\bar{n}_+(\tau_s),\ \bar{I}_+(\tau_s)) \geq F(\tau_s,\bar{n}(\tau_s),\ \bar{I}(\tau_s)) \geq \ldots \geq$$
$$\geq F(\tau_1,\bar{n}(\tau_1),\ \bar{I}(\tau_1)) \geq F(t_0,\bar{n}(t_0),\ \bar{I}(t_0))$$

Because the strategy Ω^0 is used, $\bar{I}(t_i) = Y(\bar{I}_-(t_i))$ at any time t_i. Therefore $F(t_0, \bar{n}(t_0), \bar{I}(t_0)) = F(t_0, \bar{n}(t_0), Y(\bar{I}_-(t_0))$ and $F(t_\varepsilon, 0, \bar{I}_+(t_\varepsilon)) = F(\tau_m, 0, \bar{I}_+(\tau_m))$. Taking into account that $F(\tau_m, 0, \bar{I}_+(\tau_m)) \geq F(\tau_m, \bar{n}_+(\tau_m), \bar{I}_+(\tau_m))$, $\Phi(t_0, \bar{n}(t_0), \bar{I}_-(t_0), U, \Omega^0, \Delta, \varepsilon) = F(t_\varepsilon, 0, \bar{I}_+(t_\varepsilon))$, $\bar{n}(t_0) = n(t_0)$ and $\bar{I}_-(t_0) = I_-(t_0)$, we arrive at (4.2).

Lemma 3. For any initial position $(t_0, n(t_0), I_-(t_0))$, any strategy Ω of the second player, and any Δ and ε the inequality

$$\Phi(t_0,n(t_0),I_-(t_0),U^0,\Omega,\Delta,\varepsilon) \leq$$
$$\leq \max\{F(t_0,n(t_0),Y(I_-(t_0)))+r_1\Delta,\ [r_2+r_3(\varepsilon+\Delta)](\varepsilon+\Delta)\} \tag{4.6}$$

is satisfied, where the constants r_1, r_2, r_3 depend only on the parameters of the problem.

Proof. Here by the RM we mean the motion according to the discrete scheme and the strategies U^0 and Ω. The symbols referring to the RM will be denoted by a bar, and those referring to the auxiliary StM by a tilde. The main idea is to estimate the growth of F along the RM. We specify the instants $\tau_1, \ldots, \tau_m$ when the impulses act on the RM. Since the strategy U^0 is used, the impulses are applied at those times of the discrete scheme when the centre of the RM lies either on or beyond the threshold.

A. We shall study the variation of F along the RM in the interval between τ_s and τ_{s+1}. It follows from (4) that

$$F(\tau_s,\bar{n}_+(\tau_s),\ \bar{I}_+(\tau_s)) = F(\tau_s,\bar{n}(\tau_s),\ \bar{I}(\tau_s)),\quad s=1,\ldots,m \tag{4.7}$$

We start the StM from $(\tau_s, \bar{n}_+(\tau_s), \bar{I}_+(\tau_s))$ and continue it until (τ_{s+1}). Let g_s be the number of impulses for the StM in $[\tau_s, \tau_{s+1})$. We shall establish the inequality

$$F(\tau_{s+1},\bar{n}(\tau_{s+1}),\ \bar{I}(\tau_{s+1})) \leq F(\tau_s,\bar{n}_+(\tau_s),\ \bar{I}_+(\tau_s))+\sigma g_s\Delta,\quad s=1,\ldots,m-1 \tag{4.8}$$

We denote by t' the closest time $t' \neq \tau_{s+1}$ of the discrete scheme to the left of τ_{s+1}. It is possible for t' to be equal to τ_s.

We assume that $t' > \tau_s$. In this case the centre of the RM lies below the threshold at the time t'. We estimate $\bar{\omega}^*(t')$ by $\tilde{\omega}^*(t')$. If there are no impulses of the StM in $[\tau_s, t')$, then $\bar{\omega}^*(t') \leq \tilde{\omega}^*(t')$.

We shall show that when there are impulses

$$\bar{\omega}^*(t') \leq \tilde{\omega}^*(t')+\sigma/e(\vartheta-t') \tag{4.9}$$

Let t'' be the time of the last impulse of the StM up to t'. Then $t'' > \tau_s$, the centre of the StM coincides with the threshold at t'', and $\tilde{b}(t'') = \tilde{\omega}^*_+(t'')$.

When $\tilde{b}(t'') = c$, we have

$$\bar{b}(t') \leq c \leq c\gamma(\vartheta,t'',t') = \tilde{\omega}^*_+(t'')\gamma(\vartheta,t'',t') \leq \tilde{\omega}^*(t')$$
$$\gamma(\vartheta,t'',t') = (\vartheta-t'')^2/(\vartheta-t')^2$$

Let $\tilde{b}(t'') < c$. By (2), we conclude that $\tilde{b}(t)$ evolves at the maximum possible speed in $[\tau_s, t'')$. Taking into account that $\tilde{b}(\tau_s) = \bar{b}_+(\tau_s)$ and $t'' > \tau_s$, we get $\tilde{b}(t'') \geq \bar{b}(t'')$ (even if t'' is not an instant of the discrete scheme). We have

$$\bar{b}(t') \leq \bar{b}(t'')\gamma(\vartheta,t'',t')+\zeta(t'',t') \leq \tilde{\omega}^*_+(t'')\gamma(\vartheta,t'',t')+\zeta(t'',t') = \tilde{\omega}^*(t')$$

Inequality (4.9) follows from the relationships $\bar{\omega}^*(t') = \bar{\omega}_c(t') + \bar{b}(t')$ and $\bar{\omega}_c(t') < \sigma/e(\vartheta - t')$.

Let $t' = \tau_s$. Then $\bar{\omega}^*_+(t') = \tilde{\omega}^*(t')$.

We will now prove (4.8). We set $h_s = 0$ if $t' = \tau_s$ or $t' > \tau_s$, but $[\tau_s, t')$ contains no impulses of the standard motion. We assume that $h_s = 1$ if $t' > \tau_s$ and $[\tau_s, t')$ contains some impulses of the standard motion. Let $\varphi_s \geq 0$ be the number of impulses of the standard motion in $[t', \tau_{s+1})$. With the position $(\tau_{s+1}, \bar{n}(\tau_{s+1}), \bar{I}(\tau_{s+1}))$ we associate $(\tau_{s+1}, \tilde{n}^1(\tau_{s+1}), \tilde{I}^1(\tau_{s+1}))$, where $\tilde{n}^1(\tau_{s+1}) = \bar{n}(\tau_{s+1}) + h_s + \varphi_s$, the right end $\tilde{\omega}^{1*}(\tau_{s+1})$ of $\tilde{I}^1(\tau_{s+1})$ is obtained by moving the impulses of $\tilde{\omega}^*(\tau_{s+1})$ to the right by $h_s + \varphi_s$, and the half-width $\tilde{b}^1(\tau_{s+1})$ is equal to $\tilde{b}^1(\tau_{s+1})$. We have $F(\tau_{s+1}, \tilde{n}^1(\tau_{s+1}), \tilde{I}^1(\tau_{s+1}))$ $= F(\tau_{s+1}, \tilde{n}(\tau_{s+1}), \tilde{I}^1(\tau_{s+1}))$ (briefly, $\tilde{F}^1 = \tilde{F}$).

We assume that $t' > \tau_s$. At the time t' we start the motion of the system (2.2) from $\bar{\omega}^*(t')$, with $\upsilon_z(t) \equiv \nu$ and $k(t_i) \equiv 0$. Let $\hat{\omega}(\tau_{s+1})$ be the position of the motion at the time (τ_{s+1}). We have $\bar{\omega}^*(\tau_{s+1}) \leqslant \hat{\omega}(\tau_{s+1})$. Let us equate $\bar{\omega}^*(\tau_{s+1})$ with $\tilde{\omega}^{1*}(\tau_{s+1})$. The following relations hold

$$\hat{\omega}(\tau_{s+1}) = \bar{\omega}^*(t')\gamma(\vartheta, t', \tau_{s+1}) + \zeta(t', \tau_{s+1}) \leqslant \left(\tilde{\omega}^*(t') + \frac{\sigma h_s}{e(\vartheta - t')}\right)\gamma(\vartheta, t', \tau_{s+1}) + \zeta(t', \tau_{s+1})$$

$$\tilde{\omega}^*(\tau_{s+1}) = \tilde{\omega}^*(t')\gamma(\vartheta, t', \tau_{s+1}) + \zeta(t', \tau_{s+1}) - \sum_{j=1}^{\varphi_s} \frac{\sigma(\vartheta - t_j)}{e(\vartheta - \tau_{s+1})^2}$$

$$\tilde{\omega}^{1*}(\tau_{s+1}) = \tilde{\omega}^*(\tau_{s+1}) + \frac{\sigma(h_s + \varphi_s)}{e(\vartheta - \tau_{s+1})}$$

Here t_j ($j = 1, \ldots, \varphi_s$) are the times of the impulses of the StM in the interval from t' to (τ_{s+1}). To each j there corresponds one impulse. Furthermore

$$\hat{\omega}(\tau_{s+1}) - \tilde{\omega}^{1*}(\tau_{s+1}) \leqslant \left(\tilde{\omega}^*(t') + \frac{\sigma h_s}{e(\vartheta - t')}\right)\gamma(\vartheta, t', \tau_{s+1}) - \tilde{\omega}^*(t')\gamma(\vartheta, t', \tau_{s+1}) +$$

$$+ \sum_{j=1}^{\varphi_s} \frac{\sigma(\vartheta - t_j)}{e(\vartheta - \tau_{s+1})^2} - \frac{\sigma(h_s + \varphi_s)}{e(\vartheta - \tau_{s+1})} = \frac{\sigma h_s(\vartheta - t')}{e(\vartheta - \tau_{s+1})^2} + \sum_{j=1}^{\varphi_s} \frac{\sigma(\vartheta - t_j)}{e(\vartheta - \tau_{s+1})^2} -$$

$$- \frac{\sigma(h_s + \varphi_s)}{e(\vartheta - \tau_{s+1})} \leqslant \frac{\sigma(h_s + \varphi_s)(\vartheta - t')}{e(\vartheta - \tau_{s+1})^2} - \frac{\sigma(h_s + \varphi_s)}{e(\vartheta - \tau_{s+1})} = \frac{\sigma(h_s + \varphi_s)\Delta}{e(\vartheta - \tau_{s+1})^2}$$

As a result

$$\bar{\omega}^*(\tau_{s+1}) \leqslant \tilde{\omega}^{1*}(\tau_{s+1}) + \frac{\sigma(h_s + \varphi_s)\Delta}{e(\vartheta - \tau_{s+1})^2} \tag{4.10}$$

Let $t' = \tau_s$. The estimate (4.10) also holds when $h_s = 0$. In the above argument $\bar{\omega}^*(t')$ should be replaced by $\bar{\omega}^*_+(t')$. One must also take into account that there may be several impulses at $t' = t_0$.

Let us set $\bar{F} = F(\tau_{s+1}, \bar{n}(\tau_{s+1}), \bar{I}(\tau_{s+1}))$. We shall prove the inequality

$$\bar{F} \leqslant \tilde{F}^1 + \sigma(h_s + \varphi_s)\Delta \tag{4.11}$$

The argument τ_{s+1} will be omitted for brevity.

Let $\tilde{\omega}^{1*} < \bar{\omega}^*$. We assume that $\tilde{b}^1 < \bar{b}$. Then $\tilde{\omega}^1_c < \bar{\omega}_c$. This is indeed obvious when $\tilde{b}^1 = \bar{b}$. Let $\tilde{b}^1 < \bar{b}$. If $\tilde{b} < c$, then half-width $\tilde{b}(t)$ evolves at the maximum possible speed in $[\tau_s, \tau_{s+1})$ (property 2). Since $\tilde{b}(\tau_s) = \bar{b}_+(\tau_s)$, it follows that $\tilde{b} > \bar{b}$. As a result, $\tilde{b}^1 = \tilde{b} > \bar{b}$. We have a contradiction. If $\tilde{b} = c$, then $\tilde{b}^1 = c$. Since $\tilde{\omega}^{1*} < \bar{\omega}^*$, we get $\tilde{\omega}^1_c =$ $\tilde{\omega}^{1*} - c < \bar{\omega}^* - c < \bar{\omega}_c$. The estimate (4.11) follows from Lemma 1, inequality (4.10), and the relationship $\tilde{n}^1 <$ $\bar{n}$. We assume that $\tilde{b}^1 > \bar{b}$ and set $\bar{I}^1 = (\bar{\omega}^1_c, \tilde{b}^1)$, $\bar{\omega}^1_c = \bar{\omega}^* - \tilde{b}^1$, $\bar{n}^1 = \bar{n}$, $\bar{F}^1 = F(\tau_{s+1}, \bar{n}^1, \bar{I}^1)$, and $\bar{F}^1 = F(\tau_{s+1}, \bar{n}^1, \bar{I}^1)$. Applying Lemma 1, we obtain $\bar{F}^1 \leqslant \tilde{F}^1$ when $\bar{\omega}^1_c < \tilde{\omega}^1_c$, and $\bar{F}^1 < \tilde{F}^1 + \sigma(h_s + j_s)\Delta$ when $\bar{\omega}^1_c > \tilde{\omega}^1_c$. Since $\bar{I}^1 \supset \bar{I}$, it follows that $\bar{F} \leqslant \bar{F}^1 \leqslant \tilde{F}^1 + \sigma(h_s + j_s)\Delta$.

Let $\tilde{\omega}^* > \bar{\omega}^*$. We assume that $\tilde{b}^1 < \bar{b}$. Then $\tilde{b} = \tilde{b}^1 < c$. It follows that $\tilde{b}(t)$ evolves at the maximum possible speed in $[\tau_s, \tau_{s+1})$ (property 2). Since $\tilde{b}(\tau_s) = \bar{b}_+(\tau_s)$, it follows that $\tilde{b} \geqslant b^-$. As a result, $\tilde{b}^1 = b^- > b^-$. We have a contradiction. We assume that $\tilde{b}^1 \geqslant \bar{b}$. By Lemma 1, we obtain $\tilde{F} \geqslant \bar{F}$.

Therefore, $\bar{F}$ and $\tilde{F}^1$ are by (4.11). Since $(h_s + \varphi_s) \leqslant g_s$ and $\tilde{F}^1 = \tilde{F}$, it follows from (4.11) that

$$F(\tau_{s+1}, \bar{n}(\tau_{s+1}),\ \bar{I}(\tau_{s+1})) \leqslant F(\tau_{s+1}, \tilde{n}(\tau_{s+1}),\ \tilde{I}(\tau_{s+1})) + \sigma g_s\Delta, \quad s = 1, \ldots, m-1$$

Since $F(\tau_{s+1}, \tilde{n}(\tau_{s+1}), \tilde{I}(\tau_{s+1})) = F(\tau_s, \bar{n}_+(\tau_s), \bar{I}_+(\tau_{s+1}))$ the latter implies (4.8).

By analogy with (4.8), one can prove the following inequality for $\tau_1 > t_0$ (i.e. when there is no impulse at the initial instant t_0)

$$F(\tau_1, \bar{n}(\tau_1),\ \bar{I}(\tau_1)) \leqslant F(t_0, \bar{n}(t_0),\ \bar{I}(t_0)) + \sigma g_0\Delta \tag{4.12}$$

where g_0 is the number of impulses of the StM in $[t_0, \tau_1)$ starting from the position $(t_0, \bar{n}(t_0), \bar{I}(t_0))$.

B. We consider the time $\tau_m \leqslant t_\varepsilon$ of the last impulse for the RM. First, we assume that $\bar{n}_+(\tau_m) = 0$. This means that all impulses have been spent. Using (4.7), (4.8), (4.12), and the relations $g_s \leqslant \bar{n}_+(\tau_s) \leqslant N$ and $m \leqslant N$, we obtain

$$F(\tau_m, 0, \bar{I}_+(\tau_m)) = F(\tau_m, \bar{n}_+(\tau_m),\ \bar{I}_+(\tau_m)) = F(\tau_m, \bar{n}(\tau_m),\ \bar{I}(\tau_m)) \leqslant$$

$$\leqslant F(t_0, \bar{n}(t_0),\ \bar{I}(t_0)) + \sigma N(N+1)\Delta$$

Since Ω is an arbitrary strategy, we have $F(t_i, \bar{n}(t_i), \bar{I}(t_i)) \leqslant F(t_i, \bar{n}(t_i), Y(\bar{I}_-(t_i)))$ at any time t_i by Lemma 1. Therefore $F(t_0, \bar{n}(t_0), \bar{I}(t_0)) \leqslant F(t_0, \bar{n}(t_0), Y(\bar{I}_-(t_0)))$ and $F(t_\varepsilon, 0, \bar{I}_+(t_\varepsilon)) \leqslant F(\tau_m, 0, \bar{I}_+(\tau_m))$. Since $\Phi(t_0, \bar{n}(t_0), \bar{I}_-(t_0), U^0, \Omega, \Delta, \varepsilon) = F(t_\varepsilon, 0, \bar{I}_+(t_\varepsilon))$, $\bar{n}(t_0) = n(t_0)$ and $\bar{I}_-(t_0) = I_-(t_0)$, it follows that

$$\Phi(t_0, n(t_0), I_-(t_0), U^0, \Omega, \Delta, \varepsilon) \leqslant F(t_0, n(t_0), Y(I_-(t_0))) + r_1\Delta \tag{4.13}$$

where the constant r_1 depends only on the parameters of the problem.

We assume that $\bar{n}_+(\tau_m) = 0$. Then the centre $\omega^-_{c+}(t_\varepsilon)$ of the RM lies below the threshold at t_ε. It follows that $F(t_\varepsilon, 0, \bar{I}_+(t_\varepsilon)) \leqslant [\sigma/e(\vartheta - t_\varepsilon) + c]e(\vartheta - t_\varepsilon)^2 + \nu(\vartheta - t_\varepsilon)^2/2$. Using the inequality $\vartheta - t_\varepsilon \leqslant \varepsilon + \Delta$, we get $F(t_\varepsilon, 0, \bar{I}_+(t_\varepsilon)) \leqslant [\sigma + (ce + \nu/2)(\varepsilon + \Delta)](\varepsilon + \Delta)$. As a result

$$\Phi(t_0, n(t_0), I_-(t_0), U^0, \Omega, \Delta, \varepsilon) \leqslant [r_2 + r_3(\varepsilon + \Delta)](\varepsilon + \Delta) \tag{4.14}$$

where the constants r_2 and r_3 depend only on the parameters of the problem.

By combining (4.13) and (4.14), we obtain (4.6).

Theorem. In the case when $x_0 = x^0$ the strategies U^0 and Ω^0 are optimal. Moreover

$$\Gamma^{(1)}(t_0, n(t_0), I_-(t_0)) = \Gamma^{(2)}(t_0, n(t_0), I_-(t_0)) = F(t_0, n(t_0), Y(I_-(t_0)))$$

Proof. Using (4.2) and (4.6), we get

$$L^{(2)}(t_0, n(t_0), I_-(t_0), \Omega^0) = \underline{\lim}_{\varepsilon\to 0}\, \underline{\lim}_{\Delta\to 0}\, \inf_U \Phi(t_0, n(t_0), I_-(t_0), U, \Omega^0, \Delta, \varepsilon) \geqslant$$
$$\geqslant F(t_0, n(t_0), Y(I_-(t_0)))$$
$$L^{(1)}(t_0, n(t_0), I_-(t_0), U^0) = \overline{\lim}_{\varepsilon\to 0}\, \overline{\lim}_{\Delta\to 0}\, \sup_\Omega \Phi(t_0, n(t_0), I_-(t_0), U^0, \Omega, \Delta, \varepsilon) \leqslant$$
$$\leqslant F(t_0, n(t_0), Y(I_-(t_0)))$$

It follows that

$$\Gamma^{(2)}(t_0, n(t_0), I_-(t_0)) \geqslant L^{(2)}(t_0, n(t_0), I_-(t_0), \Omega^0) \geqslant F(t_0, n(t_0), Y(I_-(t_0))) \geqslant$$
$$\geqslant L^{(1)}(t_0, n(t_0), I_-(t_0), U^0) \geqslant \Gamma^{(1)}(t_0, n(t_0), I_-(t_0)) \tag{4.15}$$

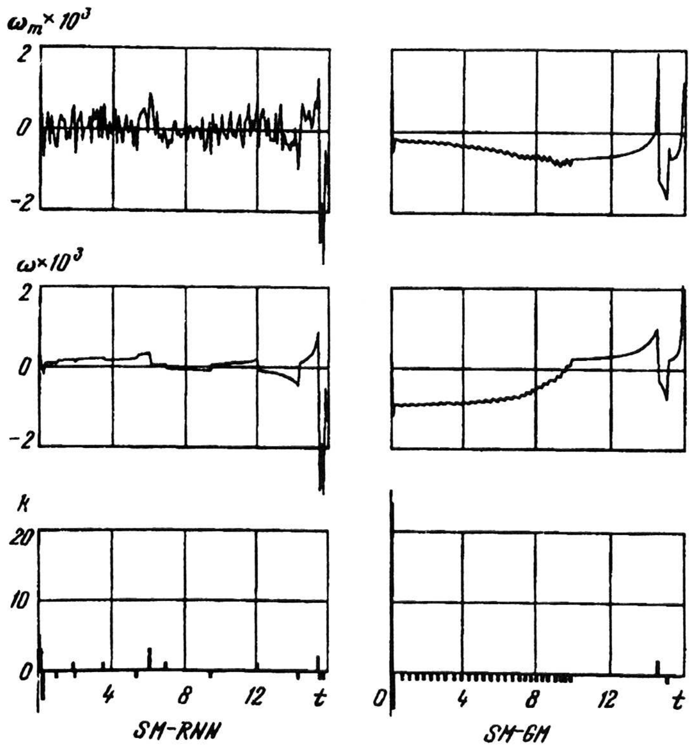
On the other hand, we have $\Gamma^{(2)}(t_0, n(t_0), I_-(t_0)) \leqslant \Gamma^{(1)}(t_0, n(t_0), I_-(t_0))$ directly from the definition of $\Gamma^{(1)}$, $\Gamma^{(2)}$. All the terms in (4.15) are therefore equal.

5. RESULTS OF MODELLING

We denote by SM the control method in (1.1) corresponding to the strategy U^0. We shall compare the SM method with the FK control method using Kalman's filtration of the angular velocity of the line of sight. The Kalman filter was programmed for a model whose dynamics is described by (2.2), the observation and control are realized with step Δ and the perturbations have a normal distribution with zero mean. The variances were given by the maximum possible deviations determined by the geometrical constraints in (2.3). The following perturbations are taken into account in the model: the dynamical miscalculation due to the acceleration υ_z, the variance being computed from the formula $(\nu/3)^2$; the error in measuring the angular velocity, the variance being taken equal to $(c/3)^2$; the initial spread of the angular velocity, the variance of which is computed as the half-width of A. The relationships of the recurrent step-by-step estimation are similar to those in [9]. Under the action of the filter, the angular velocity is estimated at each step of the discrete scheme, the impulse control in (1.1) being developed using the threshold $\sigma/x(t)$ as if this estimate where equal to the exact value.

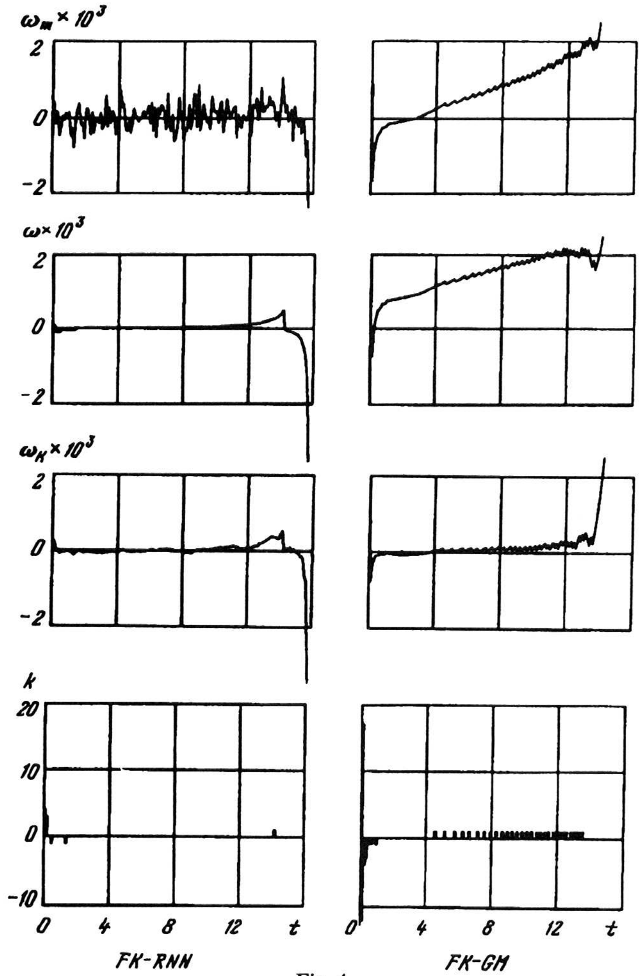
The initial values $x(t_0)$ and $V_x(t_0)$ are assumed to be known by the pursuer and equal to 80,000 and –5000 m/s, respectively. The indeterminacy of the initial horizontal position is $|z| \leqslant 2000$ m, and that of the horizontal component of the initial velocity vector is $|V_z| \leqslant 100$ m/s. The magnitude of a single impulse is $\sigma = 5$ m/s and $N = 70$ is the number of impulses. The acceleration of the evading party is $\upsilon_s \equiv 0$ vertically and $|\upsilon_z| \leqslant \nu = 2$ m/s^2 horizontally. The error in measuring the angular velocity of the line of sight is bounded by $c = 0.0009$ rad/s. The step is equal to $\Delta = 0.1$ and the threshold $\varepsilon = 0.1$. The initial information set $I_-(t_0) = A$ is an interval with end points $\omega_{-*}(t_0) = (-100 \times 80{,}000 - 5000 \times 2000)/80{,}000^2$ and $\omega_-^*(t_0) = (100 \times 80{,}000 + 5000 \times 2000)/80{,}000^2$. Since the true value of the angular velocity is computed from the formula $\omega(t) = (\dot{z}(t)x(t) - \dot{x}(t)z(t))/(x^2(t) + z^2(t))$, $I_-(t_0)$ contains all values ω that are possible because of the initial position and velocity indeterminacy.

Talking of the methods of forming the readings of $\omega_m(t_i)$ and the acceleration υ_z, we shall indicate two versions, namely, RNN and GM.



$\omega_m \times 10^3$
$\omega \times 10^3$
k
SM-RNN
SM-GM

Fig. 3.



$\omega_m \times 10^3$
$\omega \times 10^3$
$\omega_K \times 10^3$
k
FK-RNN
FK-GM

Fig. 4.

RNN. At each time t_i, the quantity $\omega_m(t_i)$ is obtained using a random number generator with a normal distribution in the interval $[\omega(t_i) - c, \omega(t_i) + c]$, where $\omega(t_i)$ is the true angular velocity of the line of sight. By analogy, the acceleration $\upsilon_z(t_i)$ is chosen from $[-\nu, \nu]$ and kept constant within Δ.

GM. In this version ω_m is produced in a similar way to the construction of Ω^0 described in Section 3, but subject to the condition that $\omega_m(t_i)$ must lie inside the interval $[\omega(t_i) - c, \omega(t_i) + c]$. We choose ω_m as follows: if ω^0 lies at the right (left) end of the interval $I_-(t_i)$, we define $\omega_m(t_i)$ so that the right (left) end of $H(\omega_m(t_i))$ lies as close as possible to this point. We choose υ_z anew at each time t_i and keep it constant in $[t_i, t_{i+1})$. If the point ω^0 computed at t_{i+1} coincides with the right end of $I_-(t_{i+1})$, then we take $\upsilon_z(t_i) = \nu$. If it coincides with the left end, then $\upsilon_z(t_i) = -\nu$.

In Fig. 3 we present the t-dependence of the observed and true angular velocities ω_m and ω realizing the impulse control for the combinations SM–RNN and SM–GM, given the initial coordinates $x(t_0) = 80{,}000$, $z(t_0) = 100$ m, $\dot{x}(t_0) = -5000$ m/s, and $\dot{z}(t_0) = 20$ m/s. The time is measured in seconds and the angular velocity in rad/s. The left column corresponds to the combination SM–RNN, the right one to SM–GM. In the first case the miscalculation amounts to 0.15 m, and 28 impulses are used. In the other case the values are 0.22 m and 58 impulses. Similar graphs for the combinations FK–RNN and FK–GM and the same initial coordinates are shown in Fig. 4. A graph for the output ω_K of the Kalman filter is added. The miscalculation for the FK–RNN combination amounts to 0.99 m, and for FK–GM it equals 51.35 m. Ten impulses were used in the former case and 70 in the latter. This means that the noise in the game makes it necessary to apply a much larger number of impulses. In the case FK–GM the entire supply of 70 impulses was used, resulting in a large miscalculation.

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