

**RUSSIAN ACADEMY OF SCIENCES**  
**Ural Branch**  
**Institute of Mathematics and Mechanics**

**Scientific report**

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**HOMICIDAL CHAUFFEUR GAME:  
HISTORY AND MODERN STUDIES**

**Ekaterinburg**  
**2009**

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**Homicidal chauffeur game: history and modern studies.**

Scientific report.

Ekaterinburg (Russia): Ural Branch of RAS, 2009. 44 p.

ISBN 978-5-7691-2041-1

“Homicidal chauffeur” game is one of the most well-known model problems in the theory of differential games. “A car” striving as soon as possible to run over “a pedestrian” – this demonstrative model suggested by R. Isaacs turned out to be appropriate for many applied problems. No less remarkable is the fact that the game is a difficult and interesting object for mathematical investigation. The report gives a survey of the literature on the homicidal chauffeur problem and its modifications. It can be of interest for students, engineers, and scientific researchers specializing in optimal control theory and applications.

Illustr. 39, refs. 33.

*Keywords:* Differential games, time-optimal control, homicidal chauffeur game, value function, numerical constructions, backward procedures

**AMS classification:** 91-02, 91A23

Responsible editor: Corresponding member of RAS A. G. Chentsov

Reviewer: Corresponding member of RAS V. N. Ushakov

The work is supported by RFBR (grants 09–01–00436, 07–01–96085) and Program of the Presidium of RAS “Mathematical control theory”.

ISBN 978-5-7691-2041-1

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## Introduction

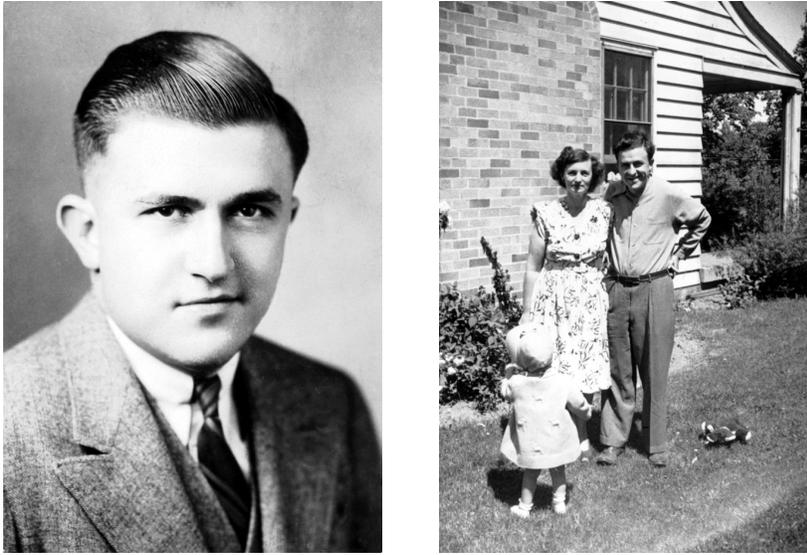
“Homicidal chauffeur” game was suggested and described by Rufus Philip Isaacs in the report [13] for the RAND Corporation in 1951. A detailed description of the problem was given in his book “Differential games” published in 1965. In this problem, a “car” whose radius of turn is bounded from below and the magnitude of the linear velocity is constant pursues a non-inertia “pedestrian” whose velocity does not exceed some given value. The names “car,” “pedestrian,” and “homicidal chauffeur” turned out to be very suitable, even if real objects that R. Isaacs meant [7, p. 543] were a guided torpedo and an evading from him small ship.

The attractiveness of the game is connected not only with its clear applied interpretation but also with the possibility of transition to reference coordinates, which enables to deal with two-dimensional state vector. In the reference coordinates, we obtain a differential game in the plane. Due to this, the analysis of the geometry of optimal trajectories and singular lines that disperse, join, or refract optimal paths becomes more transparent.

The investigation started by R. Isaacs was continued by John Valentine Breakwell and Antony Willits Merz. They improved Isaacs’ method for solving differential games and revealed new types of singular lines for problems in the plane. A systematic description of the solution structure for the homicidal chauffeur game depending on the parameters of the problem is presented in the PhD thesis by A. Merz supervised by J. Breakwell at Stanford University. The work performed by A. Merz seems to be fantastic, and his thesis, to our opinion, is the best research among those devoted to concrete model game problems.

Our report is an appreciation of the invaluable contribution made by the three outstanding scientists: R. Isaacs, J. Breakwell, and A. Merz to the differential game theory. Thanks to the help of Ellen Sara Isaacs, John Alexander Breakwell, and Antony Willits Merz we have an opportunity to present formerly unpublished photographs (Figs. 1–7).

The significance of the homicidal chauffeur game is also that it stimulated the appearance of other problems with the same dynamic equations



**Figure 1:** Left picture: Rufus Isaacs (about 1932-1936). Right picture: Rose and Rufus Isaacs with the daughter Ellen in Hartford, Connecticut before Isaacs went to Notre Dame University in about 1945.



**Figure 2:** Rose and Rufus Isaacs with their daughters Fran and Ellen on the shore of Lake Michigan in about 1947 at the time when Isaacs was at Notre Dame University.



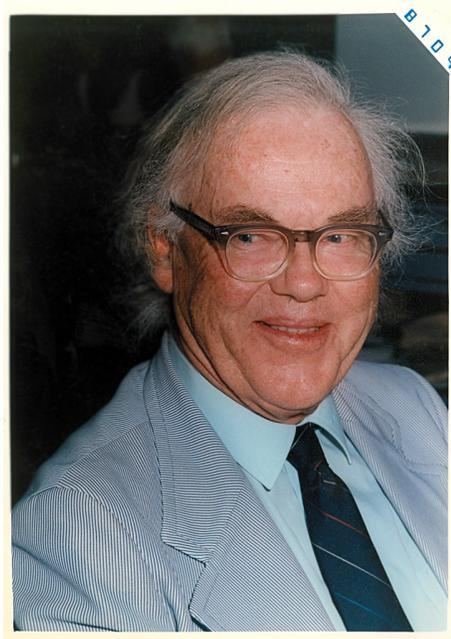
**Figure 3:** Rose and Rufus Isaacs embarking on a cruise in their 40s or 50s.



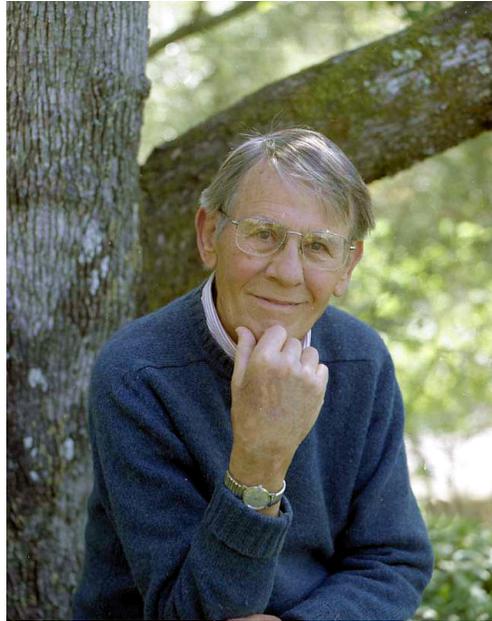
**Figure 4:** Rufus Isaacs at his retirement party, 1979.



**Figure 5:** John Breakwell at a Stanford graduation.



**Figure 6:** John Breakwell (April 1987).



**Figure 7:** Antony Merz (March 2008).

as in the classic statement, but with different objectives of the players. The most famous among them is the surveillance-evasion problem considered in papers by John Breakwell, Joseph Lewin, and Geert Jan Olsder.

Very interesting variant of the homicidal chauffeur game is investigated in the papers by Pierre Cardaliaguet, Marc Quincampoix, and Patrick Saint-Pierre. The objectives of the players are usual ones, whereas the constraint on the control of the evader depends on the distance between him and pursuer.

We also consider a statement where the pursuer is reinforced: he becomes more agile.

The description of the above mentioned problems in the paper is accompanied by the presentation of numerical results for the computation of level sets of the value function being performed using an algorithm developed by the authors. The algorithm is based on the approach for solving differential games worked out in the scientific school of Nikolai Nikolaevich Krasovskii (Ekaterinburg).

In the last section of the report, some works using the homicidal chauffeur game as a test example for computational methods are mentioned.

Also, the two-target homicidal chauffeur game is noted as a very interesting problem for the numerical investigation.

This edition is a translation of the Russian publication [30].

## 1. Classic statement by R. Isaacs

Denote the players by the letters  $P$  and  $E$ . The dynamics read

$$\begin{aligned} P : \dot{x}_p &= w \sin \theta & E : \dot{x}_e &= v_1 \\ \dot{y}_p &= w \cos \theta & \dot{y}_e &= v_2 \\ \dot{\theta} &= wu/R, |u| \leq 1 & v &= (v_1, v_2)', |v| \leq \rho. \end{aligned} \quad (1)$$

Here  $w$  is the magnitude of linear velocity,  $R$  is the minimum radius of turn. By normalizing the time and geometric coordinates, one can achieve that  $w = 1$ ,  $R = 1$ . As a result, in the dimensionless coordinates, the dynamics have the form

$$\begin{aligned} P : \dot{x}_p &= \sin \theta & E : \dot{x}_e &= v_1 \\ \dot{y}_p &= \cos \theta & \dot{y}_e &= v_2 \\ \dot{\theta} &= u, |u| \leq 1 & v &= (v_1, v_2)', |v| \leq \nu. \end{aligned} \quad (2)$$

Choosing the origin of the reference system at the position of player  $P$  and directing  $y$ -axis along  $P$ 's velocity vector, one arrives [14] at the following system

$$\begin{aligned} \dot{x} &= -yu + v_x \\ \dot{y} &= xu - 1 + v_y \\ |u| &\leq 1, v = (v_x, v_y)', |v| \leq \nu. \end{aligned} \quad (3)$$

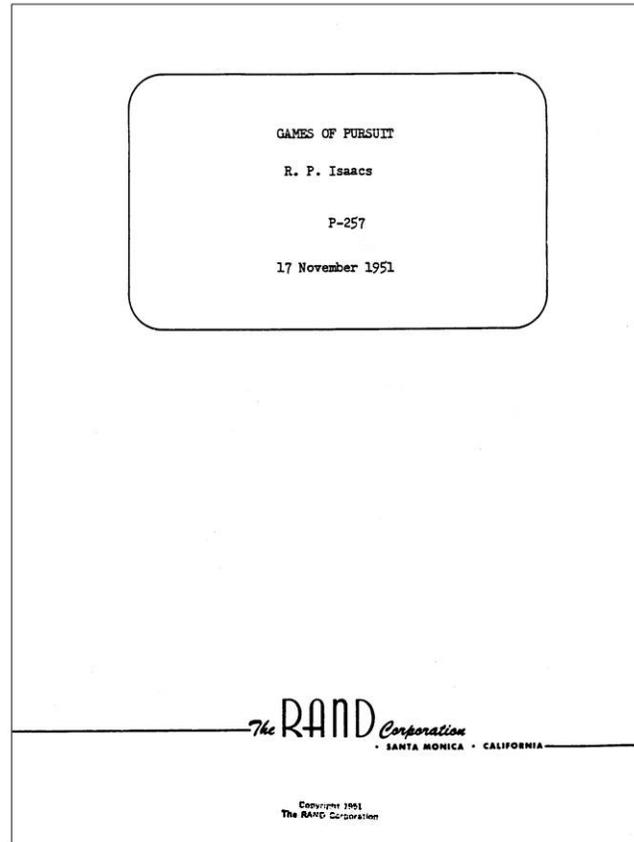
The objective of player  $P$  having control  $u$  at his disposal is, as soon as possible, to bring the state vector to the target set  $M$  being a circle of radius  $r$  with the center at the origin. The second player which steers using control  $v$  strives to prevent this. The controls are constructed based on a feedback law.

One can see that the description of the problem contains two independent parameters  $\nu$  and  $r$ .

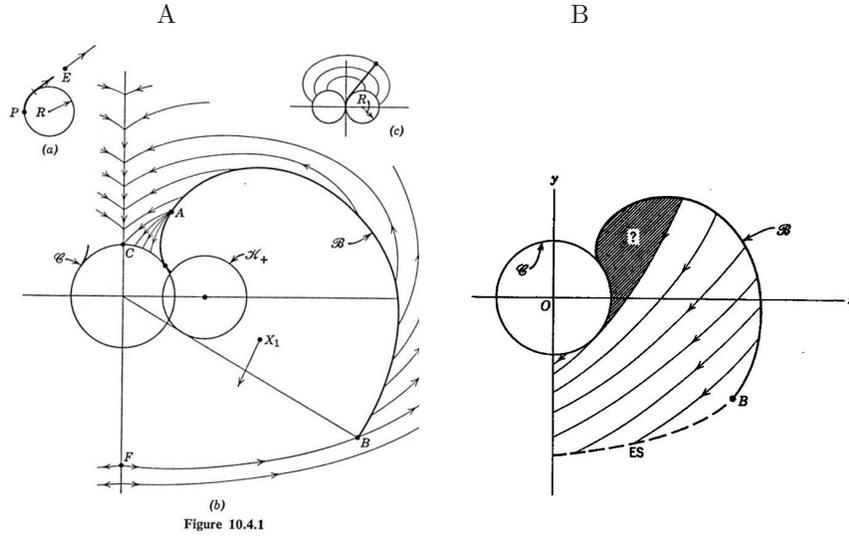
R. Isaacs investigated the problem for some parameters values using his method for solving differential games. The basis of the method is the backward computation of characteristics for an appropriate partial differential equation. First, some primary region is filled out with regular characteristics, then secondary region is filled out, and so on. The final characteristics in the plane of state variables coincide with optimal trajectories.

As it was noted, the homicidal chauffeur game was first described by R. Isaacs in his report of 1951. The title page of this report is given in Fig. 8.

Figure 9A shows a drawing from the book [14] by R. Isaacs. The solution is symmetric with respect to the vertical axis. The upper part of the vertical axis is a singular line. Forward time optimal trajectories meet this line at some angle and then go along it towards the target set  $M$ . According to the terminology by R. Isaacs, the line is called universal. The part of the vertical axis adjoining the target set from below is also a universal singular line. Optimal trajectories go down along it. The rest of the vertical axis below this universal part is dispersal: two optimal paths emanate from every point of it. On the barrier line  $\mathcal{B}$ , the value function is discontinuous. The side of the barrier line where the value of the game



**Figure 8:** Title page of the first report [13] by R. Isaacs for the RAND Corporation.



**Figure 9:** Pictures by R. Isaacs from [14] explaining the solution to the homicidal chauffeur game.

is smaller will be called positive. The opposite side is negative.

The equivocal singular line emanates tangentially from the terminal point of the barrier (Fig. 9B). It separates two regular regions. Optimal trajectories that come to the equivocal curve split into two paths: the first one goes along the curve, and the second one leaves it and comes to the regular region on the right (optimal trajectories in this region are shown in Fig. 9A).

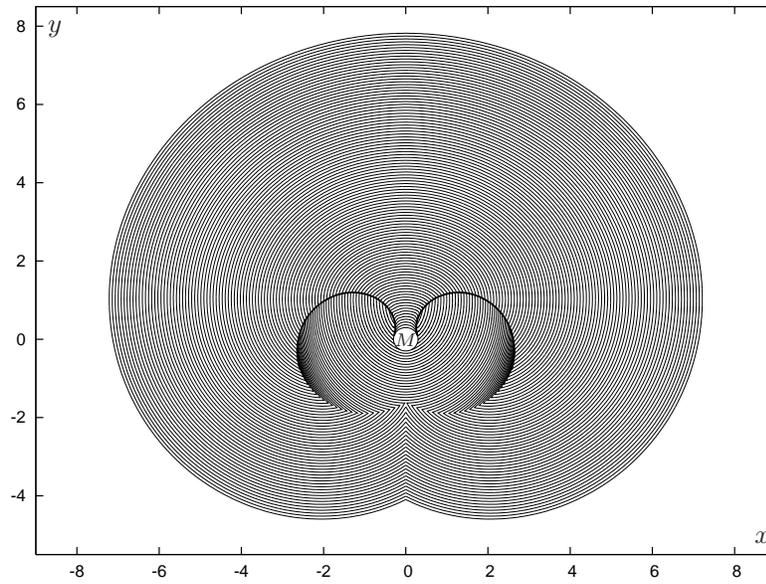
The equivocal curve is described through a differential equation which can not be integrated explicitly. Therefore, any explicit description of the value function in the region between the equivocal and barrier lines is absent. The most difficult for the investigation is the “rear” part (Fig. 9B, shaded region) denoted by R. Isaacs with a question mark. He could not obtain a solution for this region.

Figure 10 shows level sets  $W(\tau) = \{(x, y) : V(x, y) \leq \tau\}$  of the value function  $V(x, y)$  for  $\nu = 0.3, r = 0.3$ . The numerical results presented in Fig. 10 and in subsequent figures are obtained using the algorithm [29] by the authors of the paper. The lines on the boundary of the sets  $W(\tau)$ ,  $\tau > 0$ , consisting of points  $(x, y)$  where the equality  $V(x, y) = \tau$  holds, will be called fronts (isochrones). Backward construction of the fronts, beginning from the boundary of the target set, constitutes the basis of the

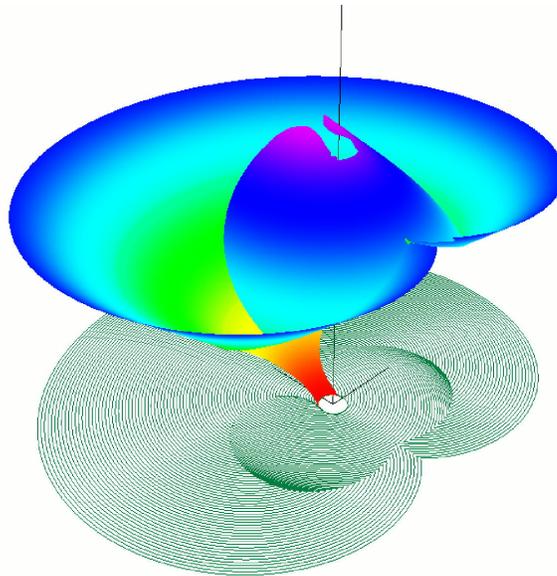
algorithm. A special computer program for the visualization of graphs of the value function in time-optimal differential games has been developed by Vladimir Lazarevich Averbukh and Oleg Aleksandrovich Pykhtev [2].

The computation for Fig. 10 is done with the time step  $\Delta = 0.01$  till the time  $\tau_f = 10.3$ . The output step for fronts is  $\delta = 0.1$ . Figure 11 presents the graph of the value function. The value function is discontinuous on the barrier lines and on a part of the boundary of the target set. In the case considered, the value function is smooth in the above mentioned rear region.

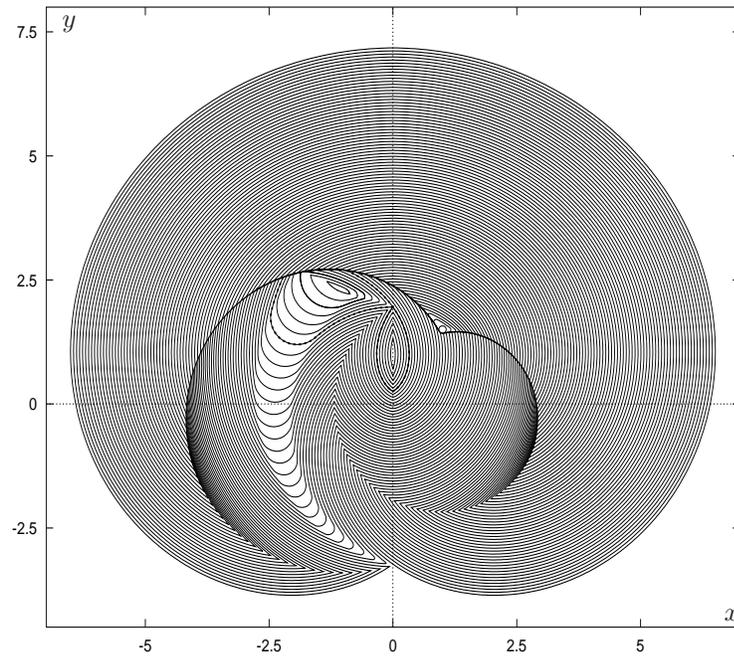
In the classical homicidal chauffeur setting, the target set is a circle centered at the origin. If the center of the target circle is shifted from the  $y$ -axis, the symmetry of the solution with respect to  $y$ -axis is destroyed. The arising front structure to the negative side of barrier lines can be very complicated. One of such examples is presented in Fig. 12. The target circle of radius 0.075 is centred at the point with the coordinates  $m_x = 1, m_y = 1.5$ . The computation time step  $\Delta = 0.01$ . The maximal value of the game for computed fronts is 9.5, and it is attained at a point in the second quadrant. The fronts are depicted with the time step  $\delta = 0.08$ . Figure 13 presents the graph of the value function.



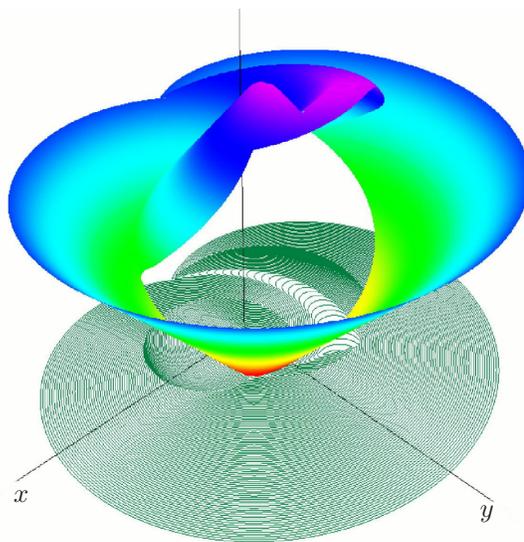
**Figure 10:** Level sets of the value function for the classical problem; game parameters  $\nu = 0.3$  and  $r = 0.3$ ; backward computation is done till the time  $\tau_f = 10.3$  with the time step  $\Delta = 0.01$ , output step for fronts  $\delta = 0.1$ .



**Figure 11:** Graph of the value function;  $\nu = 0.3$ ,  $r = 0.3$ .



**Figure 12:** Nontrivial structure of fronts for shifted target circle;  $\nu = 0.3$ ,  $\tau_f = 9.5$ ,  $\Delta = 0.01$ ,  $\delta = 0.08$ . Target set is a circle of radius  $r = 0.075$  centered at point  $(1, 1.5)$ .

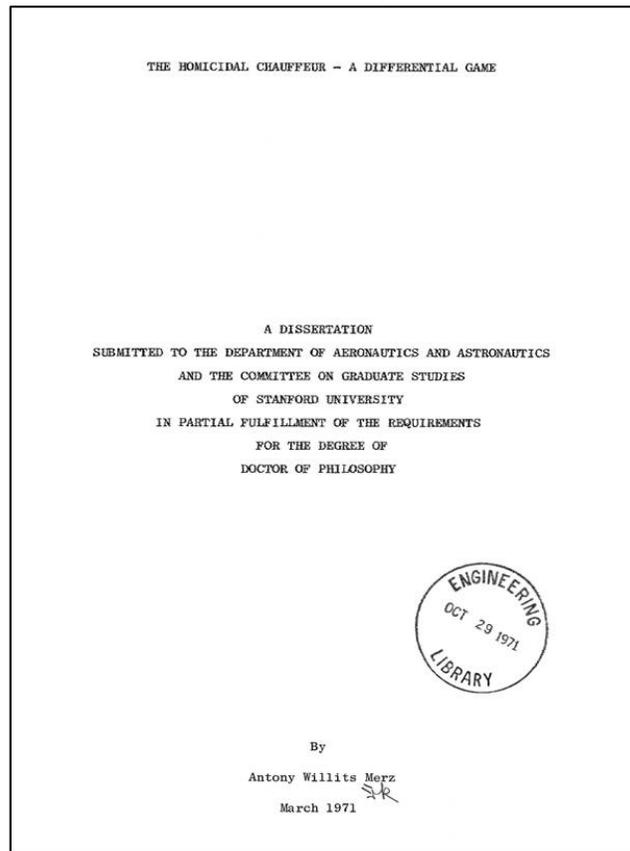


**Figure 13:** Graph of the value function for shifted target circle;  $\nu = 0.3$ ,  $r = 0.075$ .

## 2. Investigations by J. Breakwell and A. Merz

J. Breakwell and A. Merz continued investigation of the homicidal chauffeur game in the setting by R. Isaacs. Their results are partly and very briefly described in the papers [6,23]. A complete solution is obtained by A. Merz in his PhD thesis [22]. The title page of the thesis is shown in Fig. 14.

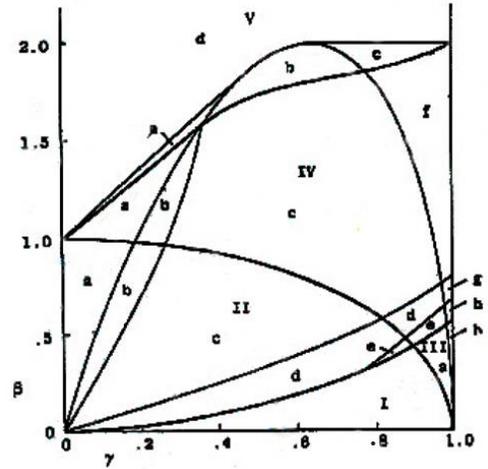
A. Merz divided two-dimensional parameter space into 20 subregions. He investigated the qualitative structure of optimal paths and the type of singular lines for every subregion. All types of singular curves (dis-



**Figure 14:** Title page of the PhD thesis by A. Merz.

persal, universal, equivocal, and switch lines) described by R. Isaacs for differential games in the plane appear in the homicidal chauffeur game for certain values of parameters. In the thesis, A. Merz suggested to distinct some subtypes of singular lines and consider them separately. Namely, he introduced the notion of focal singular lines which are universal ones, but with tangential approach of optimal paths. The value function is non-differentiable on the focal lines.

Figure 15 presents a picture and a table from the thesis by A. Merz that demonstrate the partition of two-dimensional parameter space into subregions with certain system of singular lines (A. Merz used symbols  $\gamma, \beta$  for



DISTRIBUTION OF EXCEPTIONAL LINES

Region	Sub-region	Exceptional Lines Present in Each Subregion											
		S	UL	PDL <sub>y</sub>	PDL	EDL <sub>a</sub>	EDL <sub>c</sub>	1D <sup>+</sup>	1D <sup>-</sup>	EL	SL	SE	FL
I		x	x										
II	a	x	x	x					x		x		
	b	x	x	x					x	x	x		
	c	x	x	x					x	x	x		
	d	x	x	x					x	x	x	x	
	e	x	x	x					x	x	x	x	x
III	a	x	x			x		x					
	b	x	x				x	x					
IV	a		x	x	x			x	x		x		
	b		x	x	x			x	x		x		
	c		x	x	x	x		x	x		x		
	d		x	x	x	x		x	x		x		
	e		x	x	x	x		x	x		x	x	
	f		x	x	x			x	x		x		x
	g		x	x	x			x	x		x		x
	h		x	x	x			x	x		x		x
V	a		x	x				x					
	b		x	x		x		x					
	c		x	x				x					
	d		x	x				x					

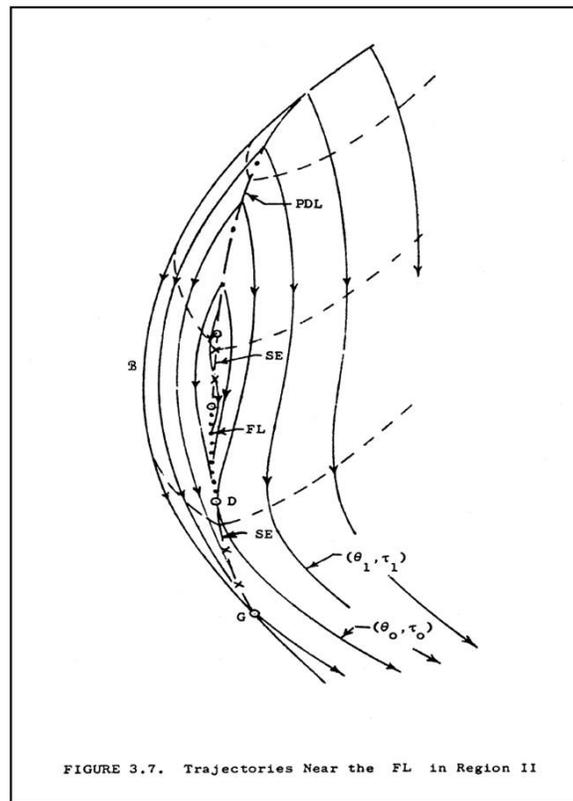
Figure 15: Decomposition of two-dimensional parameter space into subregions.

the notation of parameters. He called singular lines as exceptional lines).

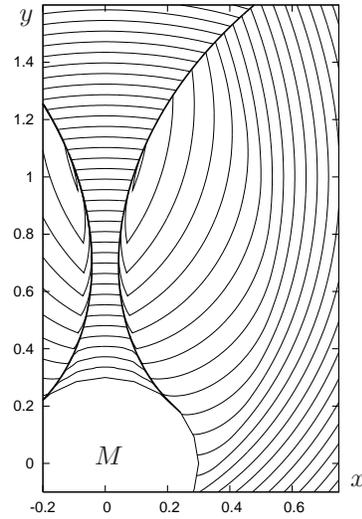
The thesis contains many pictures explaining the type of singular lines and the structure of optimal paths. By studying them, one can easily detect tendencies in the behavior of the solution depending on the change of the parameters.

In Figure 16, the structure of optimal paths in that part of the plane that adjoins the negative side of the barrier is shown for the parameters corresponding to subregion IIe. This is the rear part denoted by R. Isaacs with a question mark. For subregion IIe, very complicated situation takes place.

Symbol PDL denotes the dispersal line controlled by player  $P$ . Two optimal trajectories emanate from every point of this line. Player  $P$  controls the choice of the side to which trajectories come down. Singular curve SE



**Figure 16:** Structure of optimal paths in the rear part for subregion IIe.



**Figure 17:** Level sets of the value function for parameters from subregion IIc;  $\nu = 0.7$ ,  $r = 0.3$ ;  $\tau_f = 35.94$ ,  $\Delta = 0.006$ ,  $\delta = 0.12$ .

(the switch envelope) is specified as follows. Optimal trajectories approach it tangentially. Then one trajectory goes along this curve, and the other (equivalent) one leaves it at some angle. Therefore, line SE is similar to an equivocal singular line. The thesis contains arguments according to which the switch envelope should be better considered as an individual type of singular line.

Symbol FL denotes the focal line.

The dotted curves mark boundaries of level sets (in other words, isochrones or fronts) of the value function. The value function is not differentiable on the line composed of the curves PDL, SE, FL, and SE.

The authors of this report undertook many efforts to compute the value function for parameters from subregion IIc. But it was not succeeded, because corner points that must be present on fronts to the negative side of the barrier were absent. One of possible explanations to this failure may be the following: the effect is so subtle that it can not be detected even for very fine discretizations. The computation of level sets of the value function for the subregions where the solution structure changes very rapidly with varying parameters can be considered as a challenge for differential game numerical methods being presently developed by different scientific teams.

Figure 17 demonstrates computation results for the case where fronts have corner points in the rear region. However, the values of parameters

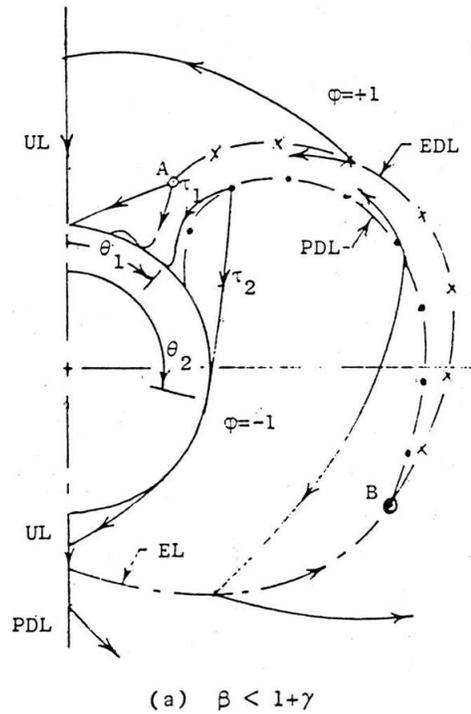


FIGURE 4.3. EDL and PDL in Region IV

**Figure 18:** Structure of optimal trajectories in subregion IVc.

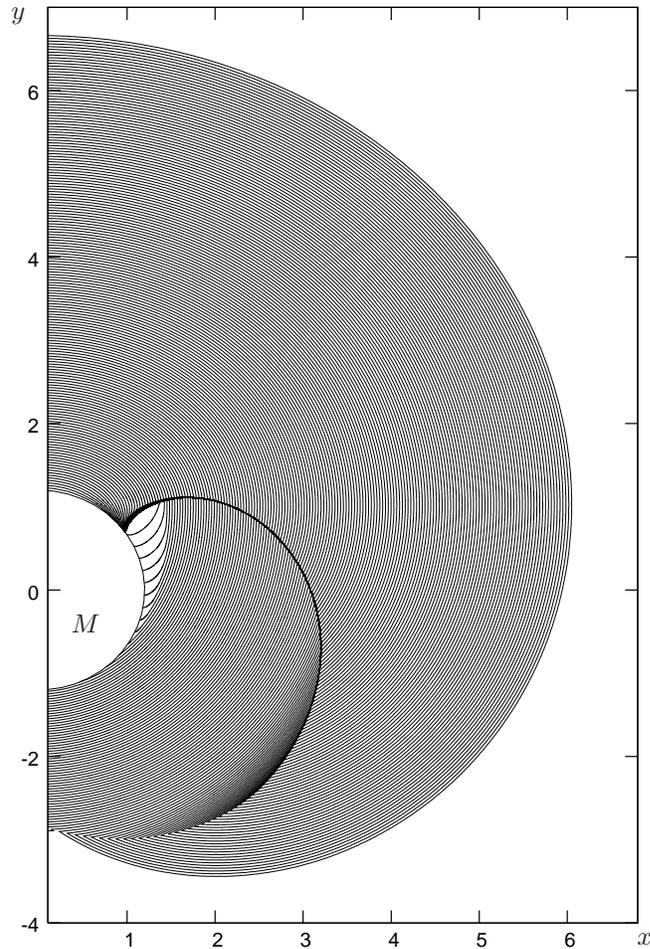
correspond not to subregion IIe but to subregion IIId. For the latter case, singular curve SE remains, whereas focal line FL disappears.

For some subregions of parameters, barrier lines on which the value function is discontinuous disappear. Figure 18 presents a picture from the thesis by A. Merz that corresponds to subregion IVc (A. Merz as well as R. Isaacs used the symbol  $\varphi$  for the notation of the control of player  $P$ . In this report, the corresponding notation is  $u$ ). For this subregion, barrier lines are absent. A. Merz described a very interesting transformation of the barrier line into two close to each other dispersal curves of players  $P$  and  $E$ . In this case, there exist both optimal paths that go up and those that go down along the boundary of the target set. The investigation of such a phenomenon is of great theoretical interest.

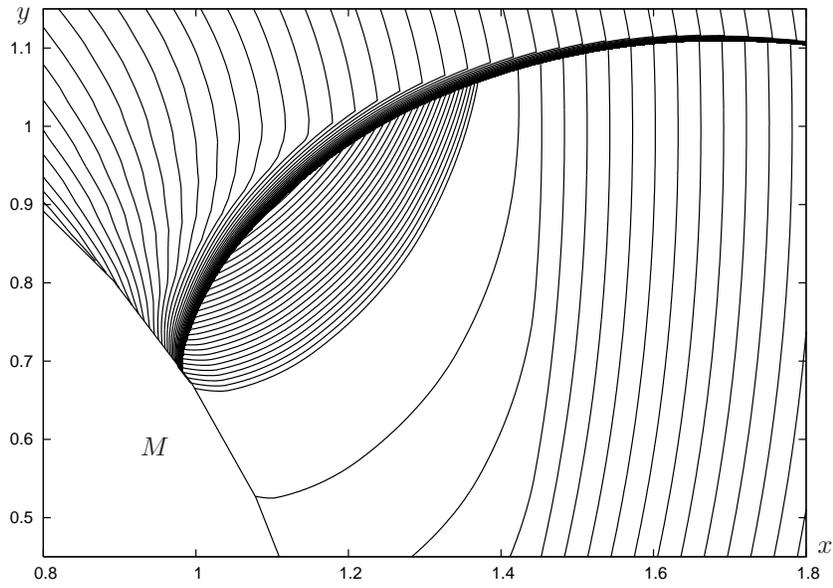
Numerically constructed level sets of the value function for the magnitudes of parameters  $\nu$ ,  $r$  from subregion IVc are shown in Fig. 19. When examining Fig. 19, it might seem that some barrier line exists. But this is

not true. We have exactly the case like the one shown in Fig. 18.

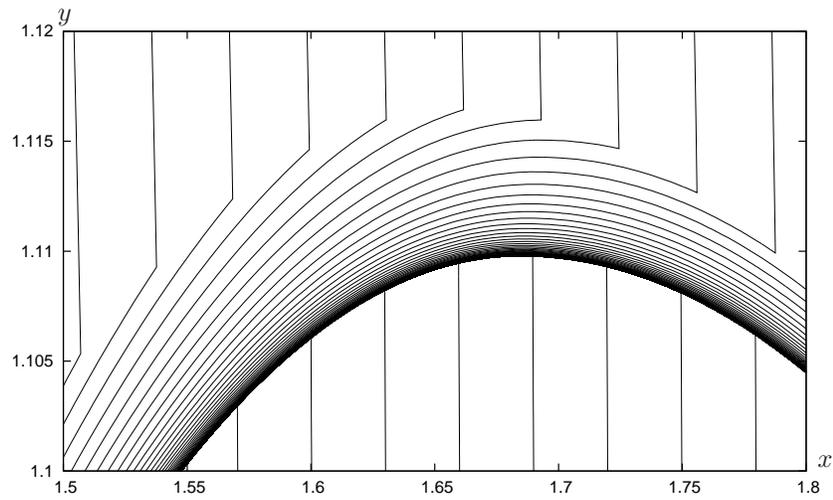
Enlarged fragments of numerical constructions are given in Figs. 20, 21 (the scale of  $y$ -axis in Fig. 21 is increased with respect to that of  $x$ -axis). The curve consisting of fronts' corner points above the accumulation region of fronts is the dispersal line of player  $E$ . The curve composed of corner points below the accumulation region is the dispersal line of player  $P$ . The value function is continuous in the accumulation region. To see where (in the considered part of the plane) the point with a maximum value of the game is located, additional fronts are shown. The point with the maximum



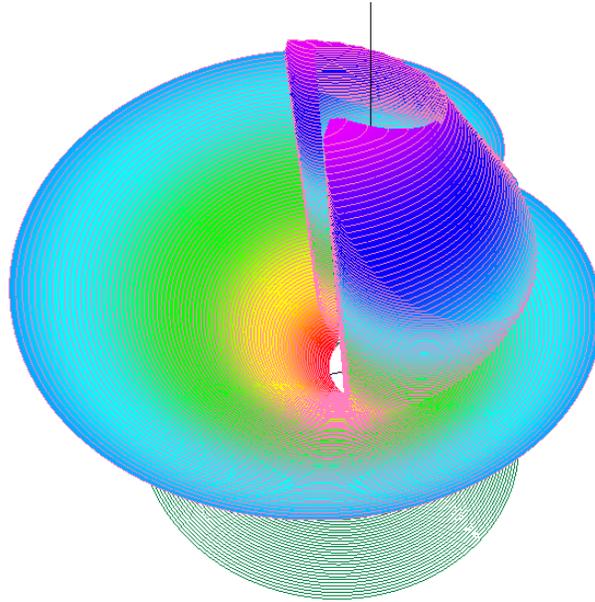
**Figure 19:** Level sets of the value function;  $\nu = 0.7$ ,  $r = 1.2$ ;  $\tau_f = 24.22$ ,  $\Delta = 0.005$ ,  $\delta = 0.1$ .



**Figure 20:** Enlarged fragment of Fig. 19;  $\tau_f = 24.22$ . Output step for fronts close to the time  $\tau_f$  is decreased up to  $\delta = 0.005$ .



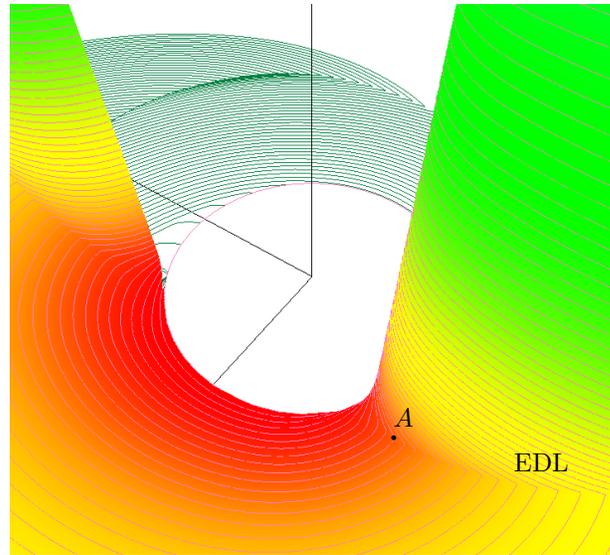
**Figure 21:** Enlarged fragment of Fig. 19.



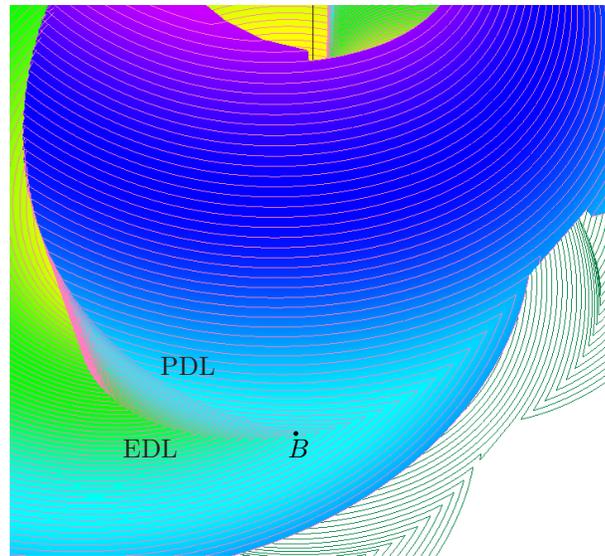
**Figure 22:** Graph of the value function;  $\nu = 0.7$ ,  $r = 1.2$ . Level lines are plotted. Salient curve corresponding to the line PDL from Fig. 18 is seen.

value has coordinates  $x = 1.1$ ,  $y = 0.92$ . The value function at this point is equal to 24.22.

The graph of the value function for the example considered is shown in Fig 22. The level lines are plotted to make visible two curves consisting of salient points. Taking into account the symmetry with respect to the  $y$ -axis, the curves correspond to the dispersal singular lines EDL and PDL in the plane  $x, y$  (see Fig. 18). Preserve the notation EDL and PDL for the the salient curves from the graph of the value function. The perspective and scale of Fig. 23 are chosen in such a way that the part of the graph where the curve EDL arises is well seen. The origin of the curve is denoted by the letter  $A$ . In Fig. 24, the part of the graph near the point  $B$  where the curve EDL terminates (the value of the game increases when traveling along EDL from the point  $A$  to the point  $B$ ) and, simultaneously, the curve PDL starts is shown. The salient curve which smoothly continues the curve EDL (with the increase of the game value) is not anymore dispersal. As it is specified in Fig. 18, it represents an equivocal singular line.



**Figure 23:** Fragment of the graph of the value function for the same parameters as in Fig. 22. The salient curve corresponding to the line EDL from Fig. 18 starts at the point  $A$ .



**Figure 24:** Fragment of the graph of the value function for the same parameters as in Fig. 22. The salient curve corresponding to the line PDL from Fig. 18 starts at the point  $B$ .

### 3. Surveillance-evasion game

In the PhD thesis by J. Lewin [18] (performed as well under the supervision of J. Breakwell), in the joint paper by J. Breakwell and J. Lewin [19], and also in the paper by J. Lewin and G. J. Olsder [20], both dynamics and constraints on the controls of the players are the same as in Isaacs' setting but the objectives of the players differ from those in the classic statement. Namely, player  $E$  tries to decrease the time of reaching the target set  $M$  by the state vector, whereas player  $P$  strives to increase that time. In the first and second works, the target set is the complement (with respect to the plane) of an open circle centered at the origin. In the third publication, the target set is the complement of an open cone with the apex at the origin.

The meaning related to the original context concerning two moving vehicles is the following: player  $E$  tries, as soon as possible, to escape from some detection zone attached to the geometric position of player  $P$ , whereas player  $P$  strives to keep his opponent in the detection zone as long as possible. Such a problem was called the surveillance-evasion game. To solve it, J. Breakwell, J. Lewin, and G. J. Olsder used Isaacs' method.

One picture from the thesis by J. Lewin is shown in Fig. 25A, and one picture from the paper by J. Lewin and G. J. Olsder is given in Fig. 25B.

In the surveillance-evasion game with the conic target set, examples of transition from finite values of the game to infinite values are of interest and can be easily constructed.

Figure 26 shows level sets of the value function for five values of parameter  $\alpha$  which specifies the semi-angle of the nonconvex conic detection zone. Since the solution to the problem is symmetric with respect to  $y$ -axis, only the right half-plane is shown for four of five figures. The pictures are ordered from greater to smaller  $\alpha$ .

In the first picture, the value function is finite in the set that adjoins the target cone and is bounded by the curve  $a'b'cba$ . This set is filled out with the fronts (isochrones). The value function is zero within the target set. Outside the union of the target set and the set filled out with the fronts, the value function is infinite.

In the third picture, a situation of the accumulation of fronts is presented. Here, the value function is infinite on the line  $fe$  and finite on the arc  $ea$ . The value function has a finite discontinuity on the arc  $be$ . The graph of the value function corresponding to the third picture is shown in Fig. 27A.

A

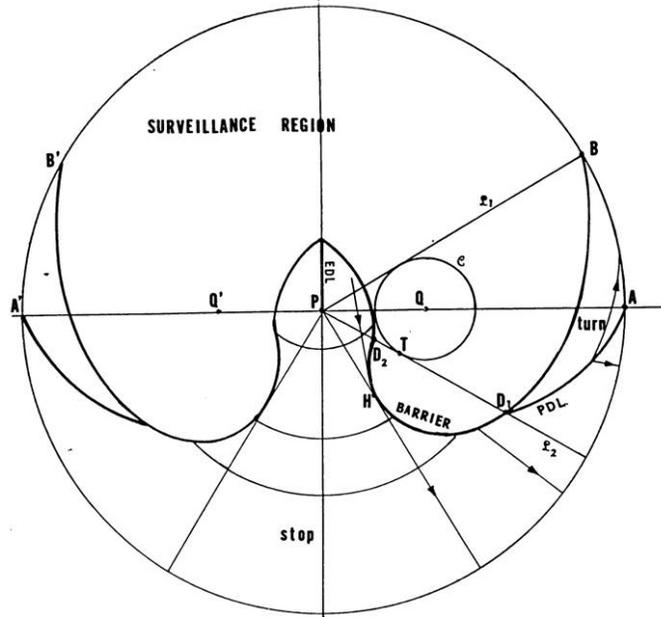


Fig. 5.21  $f_2(\psi) < \beta \leq f_3(\psi)$

B

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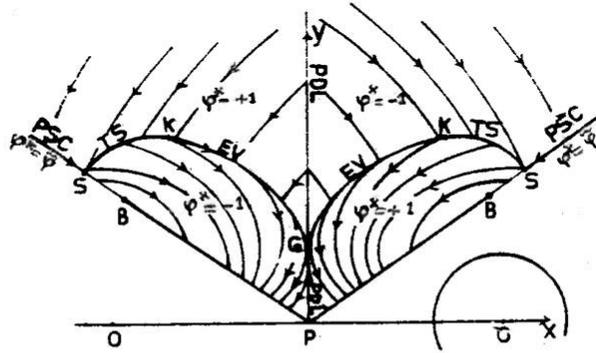
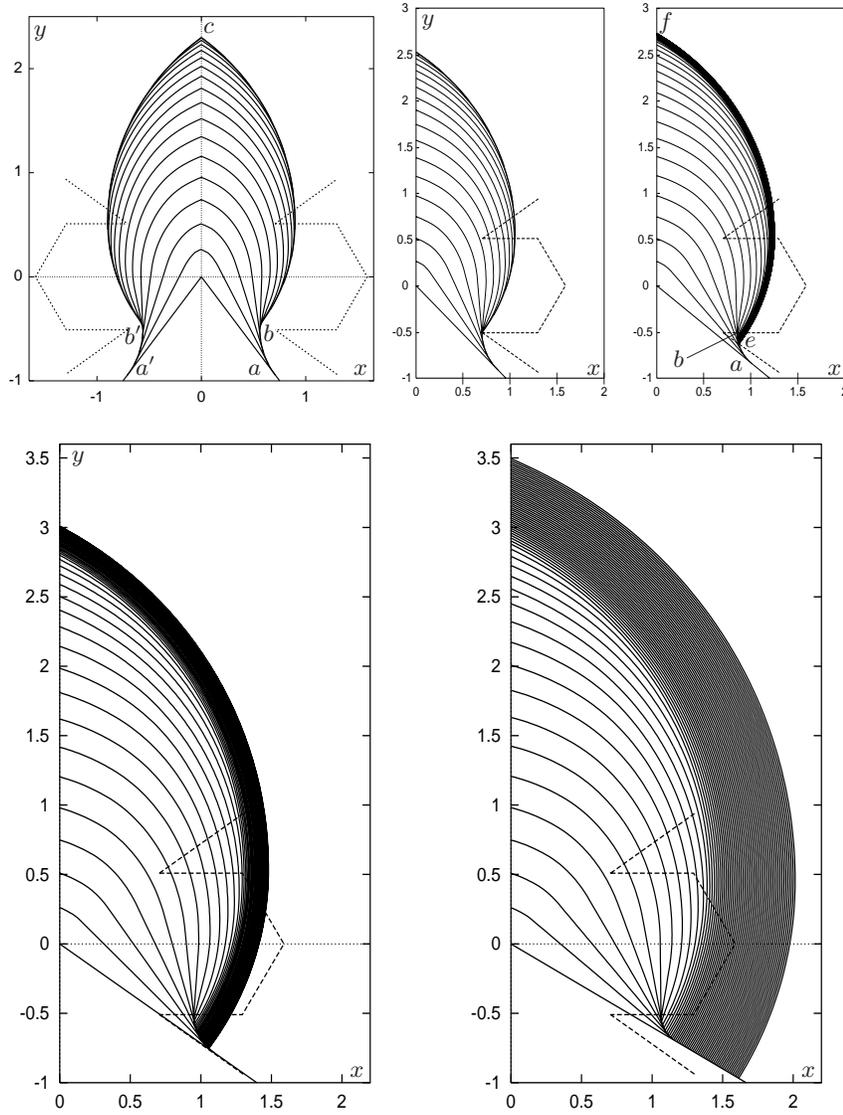


Fig. 8.  $\pi/2 \geq \theta \geq \pi/4$ .

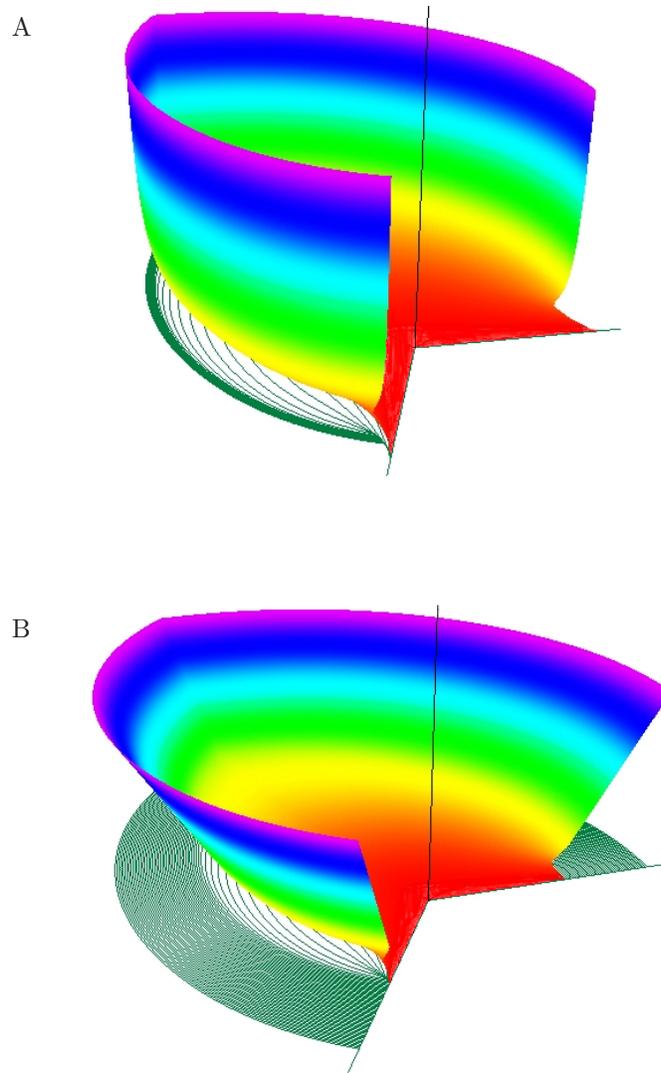
**Figure 25:** (A) Picture from the PhD thesis by J. Lewin. Detection zone is a circle. (B) Picture from the paper by J. Lewin and G. J. Olsder. Detection zone is a convex cone.



**Figure 26:** Surveillance-evasion game. Change of the front structure depending on the semi-angle  $\alpha$  of the nonconvex detection cone;  $\nu = 0.588$ ,  $\Delta = 0.017$ ,  $\delta = 0.17$ .

The second picture demonstrates a transition case from the first to the third picture.

In the fifth picture, the fronts propagate slowly to the right and fill out



**Figure 27:** Value function in the surveillance-evasion game. (A)  $\nu = 0.588$ ,  $\alpha = 130^\circ$ , (B)  $\nu = 0.588$ ,  $\alpha = 121^\circ$ .

(outside the target set) the right half-plane as the backward time  $\tau$  goes to infinity. Figure 27B gives a graph of the value function for this case.

The fourth picture shows a transition case between the third and fifth pictures.

#### 4. Acoustic version of the homicidal chauffeur game

Let us return to problems where player  $P$  minimizes and player  $E$  maximizes the time of reaching the target set  $M$ . In papers [8,9], P. Cardaliaguet, M. Quincampoix, and P. Saint-Pierre have considered an “acoustic” variant of the homicidal chauffeur problem suggested by Pierre Bernhard [4]. It is supposed that the constraint  $\nu$  on the control of player  $E$  depends on the state  $(x, y)$ . Namely,

$$\nu(x, y) = \nu^* \min \left\{ 1, \sqrt{x^2 + y^2}/s \right\}, \quad s > 0.$$

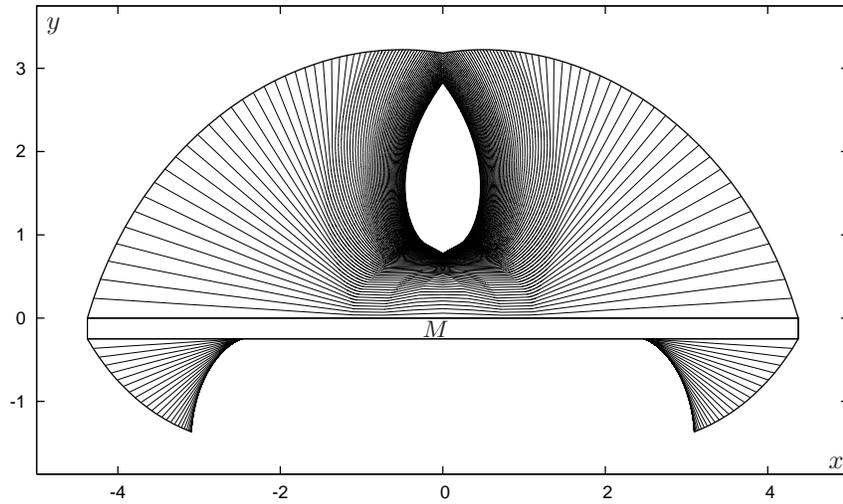
Here,  $\nu^*$  and  $s$  are the parameters of the problem.

The applied aspect of the acoustic game: object  $E$  should not be very loud if the distance between him and object  $P$  becomes less than a given value  $s$ .

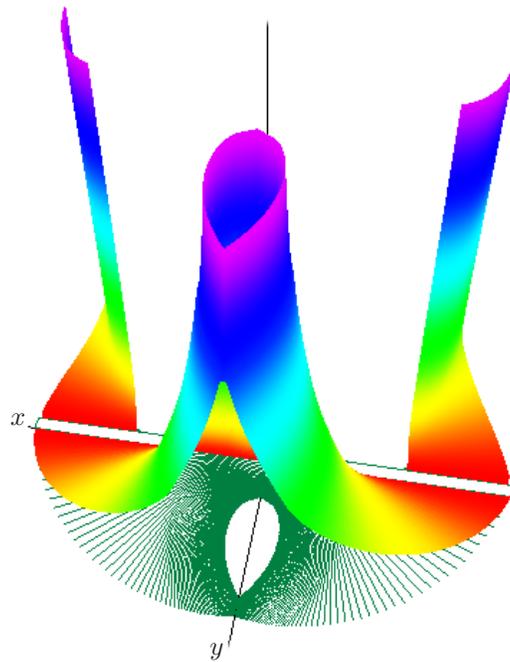
P. Cardaliaguet, M. Quincampoix, and P. Saint-Pierre investigated the acoustic problem using their own method for numerical solving of differential games which goes back to the viability theory [1]. It was revealed that one can choose the values of parameters in such a way that the set of states where the value function is finite will contain a hole in which points the value function is infinite. Especially easy such a case can be obtained when the target set is a rectangle stretched along the horizontal axis.

Figures 28 and 29 demonstrate an example of the acoustic problem with the hole. The level sets of the value function and the graph of the value function are shown. The value of the game is infinite outside the set filled out with the fronts. An exact theoretical description of the arising hole and the computation (both analytical and numerical) of the value function near the boundary of the hole seems to be very complicated problem.

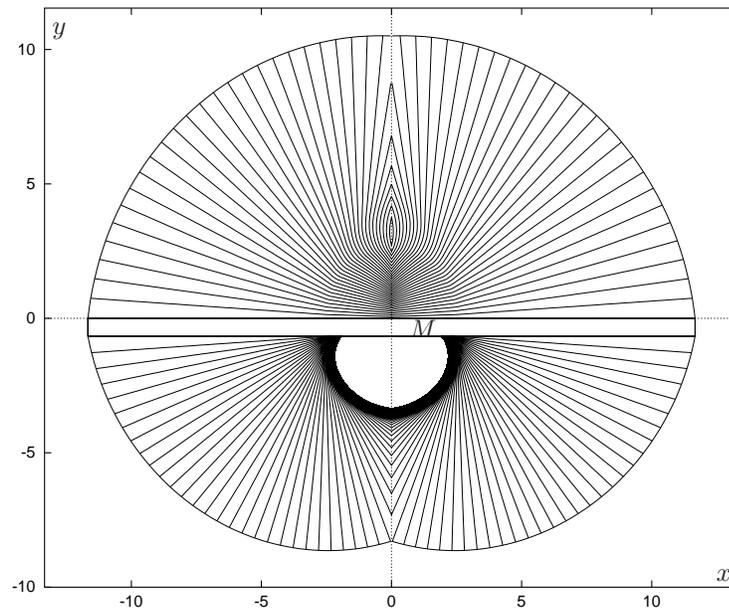
Let us underline that the above mentioned hole is separated from the target set. In Figure 30, level sets for the parameters  $\nu^* = 1.4$ ,  $s = 2.5$  are presented. The graph of the value function is shown in Fig. 31. Also here, a hole with infinite magnitudes of the value function arises. But this hole touches the target set, which allows one to compute it easily through the barrier lines emanated from some points on the boundary of the target set.



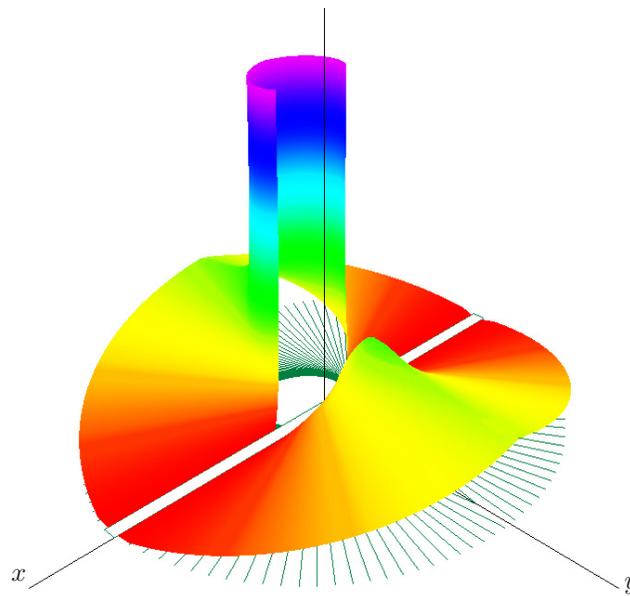
**Figure 28:** Level sets of the value function in the acoustic problem;  $\nu^* = 1.5$ ,  $s = 0.9375$ ;  $\Delta = 0.00625$ ,  $\delta = 0.0625$ .



**Figure 29:** Graph of the value function in the acoustic problem;  $\nu^* = 1.5$ ,  $s = 0.9375$ .



**Figure 30:** Level sets of the value function in the acoustic problem;  $\nu^* = 1.4$ ,  $s = 2.5$ ;  $\Delta = 1/30$ ,  $\delta = 1/6$ .



**Figure 31:** Graph of the value function in the acoustic problem;  $\nu^* = 1.4$ ,  $s = 2.5$ .

## 5. Game with a more agile player $P$

The model of dynamics of player  $P$  in Isaacs' setting is the simplest one among those used in mathematical publications for the description of the car motion (or the aircraft motion in the horizontal plane). In this model, the trajectories are curves of bounded curvature. In the paper [21] by Andrey Andreevich Markov published in 1889, four problems related to the optimization over the curves with bounded curvature have been considered. The first problem (Fig. 32) can be interpreted as a time-optimal control problem where a car has the dynamics of player  $P$ . Similar interpretation can be given to the main theorem (Fig. 33) of the paper [11] by Lester E. Dubins published in 1957. The name "car" is not used neither by A. A. Markov, nor by L. Dubins. A. A. Markov mentioned problems of railway construction. In modern works on theoretical robotics [17], an object with the classical dynamics of player  $P$  is called "Dubins' car."

The next in complexity is the car model from the paper by James A. Reeds and Lawrence A. Shepp [33]:

$$\begin{aligned} \dot{x}_p &= w \sin \theta \\ \dot{y}_p &= w \cos \theta \\ \dot{\theta} &= u, \quad |u| \leq 1, \quad |w| \leq 1. \end{aligned}$$

The control  $u$  determines the angular velocity of motion. The control  $w$  is responsible for the instantaneous change of the linear velocity magnitude. In particular, the car can instantaneously reverse the direction of motion. A non-inertia change of the linear velocity magnitude is a mathematical idealization. But, citing [33, p. 373], "for slowly moving vehicles, such as carts, this seems like a reasonable compromise to achieve tractability."

It is natural to consider problems where the range for changing the control  $w$  is  $[a, 1]$ . Here,  $a \in [-1, 1]$  is the parameter of the problem. If  $a = 1$ , Dubins' car is obtained. For  $a = -1$ , one arrives at Reeds-Shepp's car.

Let us replace in (2) the classic car by a more agile car. Using the transformation to the reference coordinates, we obtain

$$\begin{aligned} \dot{x} &= -yu + v_x \\ \dot{y} &= xu - w + v_y \end{aligned} \tag{4}$$

$|u| \leq 1, \quad w \in [a, 1], \quad v = (v_x, v_y)', \quad |v| \leq \nu.$

Player  $P$  is responsible for the controls  $u$  and  $w$ , player  $E$  steers with the control  $v$ .

## Нѣсколько примѣровъ рѣшенія особаго рода задачъ о наибольшихъ и наимень- шихъ величинахъ.

А. А. Маркова.

### ЗАДАЧА 1.

Между данными точками  $A$  и  $B$  (см. фиг. 1-ю) провести кратчайшую кривую линію при слѣдующихъ двухъ условіяхъ: 1) радиусъ кривизны нашей кривой повсюду долженъ быть не меньше данной величины  $\rho$ , 2) въ точкѣ  $A$  касательная къ нашей кривой должна имѣть данное направленіе  $AC$ .

### РѢШЕНІЕ.

Пусть  $M$  одна изъ точекъ нашей кривой, а прямая  $NMT$  соответствующая касательная.

**Figure 32:** Fragment of the first page of the paper by A. Markov “Some examples of the solution of a special kind of problem on greatest and least quantities.” “Problem 1: Find a minimum length curve between given points  $A$  and  $B$  provided that the following conditions are satisfied: 1) the curvature radius of the curve should not be less than a given quantity  $\rho$  everywhere, 2) the tangent to the curve at point  $A$  should have a given direction  $AC$ . Solution: Let  $M$  be a point of our curve, and the straight line  $NMT$  be the corresponding tangent...”

Note that J. Breakwell and J. Lewin investigated the surveillance-evasion game [18,19] with the circular detection zone in the assumption that, at every time instant, player  $P$  either moves with the unit linear velocity or remains immovable. Therefore, they actually considered dynamics like (4) with  $a = 0$ .

The homicidal chauffeur game where player  $P$  controls the car which is able to change his linear velocity magnitude instantaneously was considered by the authors of this paper in [31]. The dependence of the solution on the parameter  $a$  specifying the left end of the constraint to the linear velocity magnitude was investigated numerically.

In Figure 34, the level sets of the value function which correspond to one and the same time  $\tau = 3$  but to different values of the parameter  $a$  from  $-1$  to  $1$  are presented. For all computations, the radius of the target set is  $r = 0.3$  and the constraint on the control of player  $E$  is  $\nu = 0.3$ . In case  $a = -1$ , player  $P$  controls Reeds-Shepp’s car, and the obtained level set is symmetric with respect to both  $y$ -axis and  $x$ -axis. If  $a = 1$ , the level set for the classical homicidal chauffeur game is obtained.

AMERICAN  
JOURNAL OF MATHEMATICS

FOUNDED BY THE JOHNS HOPKINS UNIVERSITY

Volume LXXIX, Number 3  
JULY, 1957

ON CURVES OF MINIMAL LENGTH WITH A CONSTRAINT ON  
AVERAGE CURVATURE, AND WITH PRESCRIBED INITIAL  
AND TERMINAL POSITIONS AND TANGENTS.\*<sup>1</sup>

By L. E. DUBINS.

We have now established our main result:

**THEOREM I.** *Every planar  $R$ -geodesic is necessarily a continuously differentiable curve which is either (1) an arc of a circle of radius  $R$ , followed by a line segment, followed by an arc of a circle of radius  $R$ ; or (2) a sequence of three arcs of circles of radius  $R$ ; or (3) a subpath of a path of type (1) or (2).*

Figure 33: Two fragments of the paper by L. Dubins.

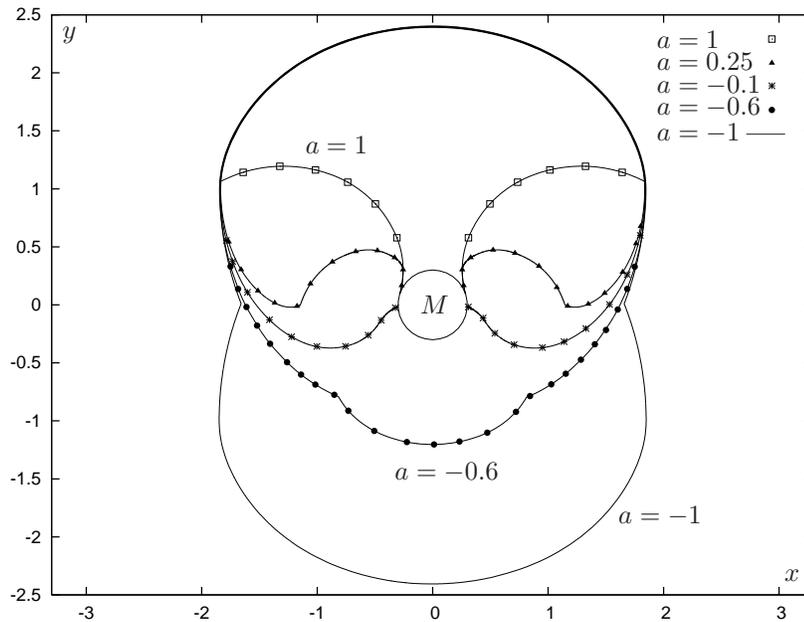
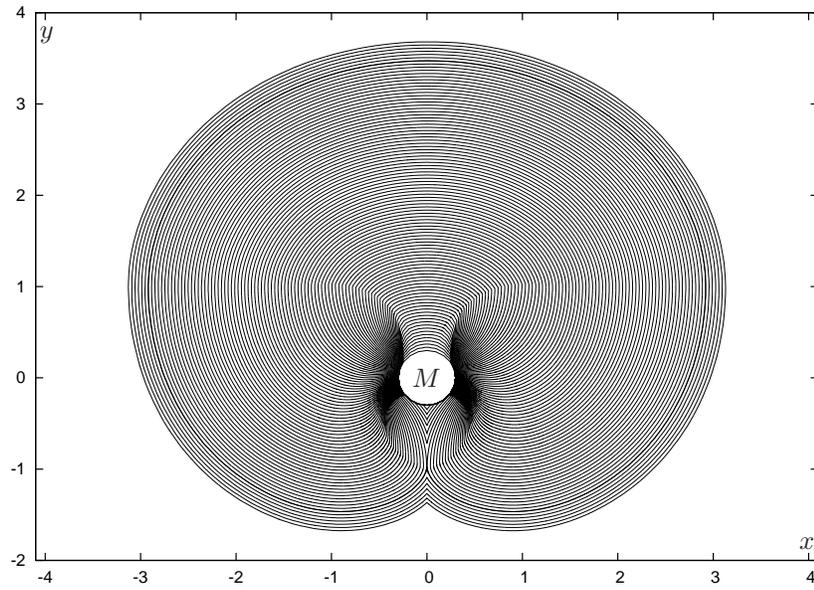
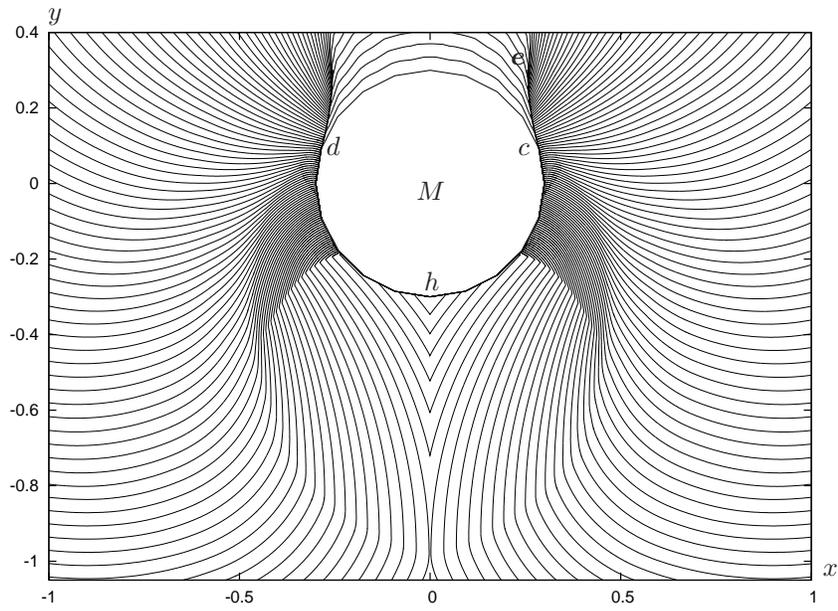


Figure 34: Homicidal chauffeur game with more agile pursuer. Dependence of level sets of the value function on the parameter  $a$  for  $\tau = 3$ ;  $\nu = 0.3$ ,  $r = 0.3$ .

Figure 35 shows the level sets of the value function for  $a = -0.1$ ,  $\nu = 0.3$ ,  $r = 0.3$ . The computation is done backward in time till  $\tau_f = 4.89$ . Precisely this value of the game corresponds to the last outer front and to the last inner front adjoining to the lower part of the boundary of the target circle  $M$ . The front structure is well visible in Fig. 36 showing an enlarged fragment of Fig. 35. One can see a nontrivial character of changing the fronts near the lower border of the accumulation region. The value function is discontinuous on the arc  $dhc$ . It is also discontinuous outside  $M$  on two short barrier lines emanating tangentially from the boundary of  $M$ . The right barrier is denoted by  $ce$ .



**Figure 35:** Level sets of the value function in the homicidal chauffeur game with more agile pursuer;  $a = -0.1$ ,  $\nu = 0.3$ ,  $r = 0.3$ ;  $\tau_f = 4.89$ ,  $\Delta = 0.002$ ,  $\delta = 0.05$ .



**Figure 36:** Enlarged fragment of Fig. 35.

## 6. Optimal strategies

When solving time-optimal differential games of the homicidal chauffeur type (with discontinuous value function), the most difficult task is the construction of optimal (or  $\varepsilon$ -optimal) strategies of the players. Let us demonstrate such a construction using the last example of the previous section.

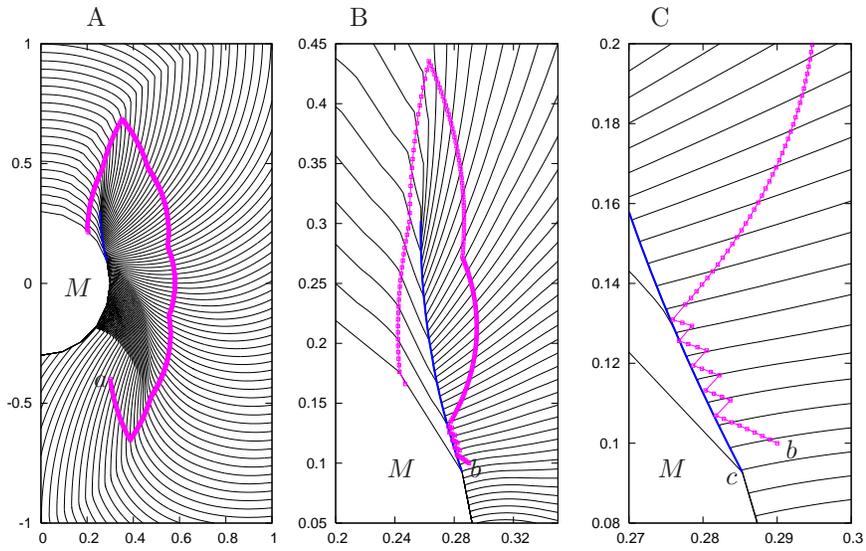
We construct  $\varepsilon$ -optimal strategies using the extremal aiming procedure [15,16]. The computed control remains unchanged during the next step of the discrete control scheme. The step of the control procedure is a modeling parameter. The strategy of player  $P$  ( $E$ ) is defined using the extremal shift to the nearest point (extremal repulsion from the nearest point) of the corresponding front. If the trajectory comes to a prescribed layer attached to the positive (negative) side of the discontinuity line of the value function, then a control which pushes away from the discontinuity line is utilized.

Let us choose two initial points  $a = (0.3, -0.4)$  and  $b = (0.29, 0.1)$ . The first point is located in the right half-plane below the front accumulation region, the second one is close to the barrier line on its negative side. The values of the game in the points  $a$  and  $b$  are  $V(a) = 4.225$  and  $V(b) = 1.918$ , respectively.

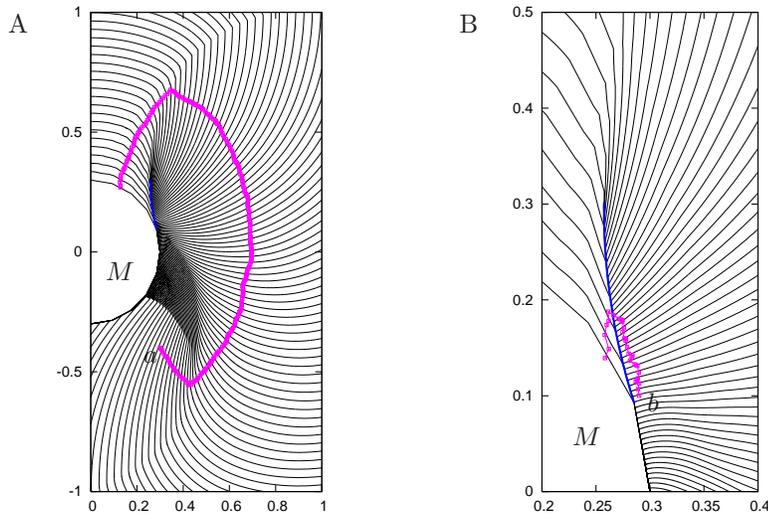
In Figure 37, the trajectories for  $\varepsilon$ -optimal strategies of the players are shown. The time step of the control procedure is 0.01. We obtain that the time of reaching the target set  $M$  is equal to 4.230 for the point  $a$  and 1.860 for the point  $b$ . Figure 37C demonstrates an enlarged fragment of the trajectory emanating from the initial point  $b$ . One can see a sliding mode along the negative side of the barrier.

Figure 38 presents trajectories for non-optimal behavior of player  $E$  and optimal behavior of player  $P$ . The control of player  $E$  is computed using a random number generator (random choice of vertices of the polygon approximating the circle constraint of player  $E$ ). The reaching time is 2.590 for the point  $a$  and 0.300 for the point  $b$ . One can see how the second trajectory penetrates the barrier line. In this case, the value of the game calculated along the trajectory drops jump-wise.

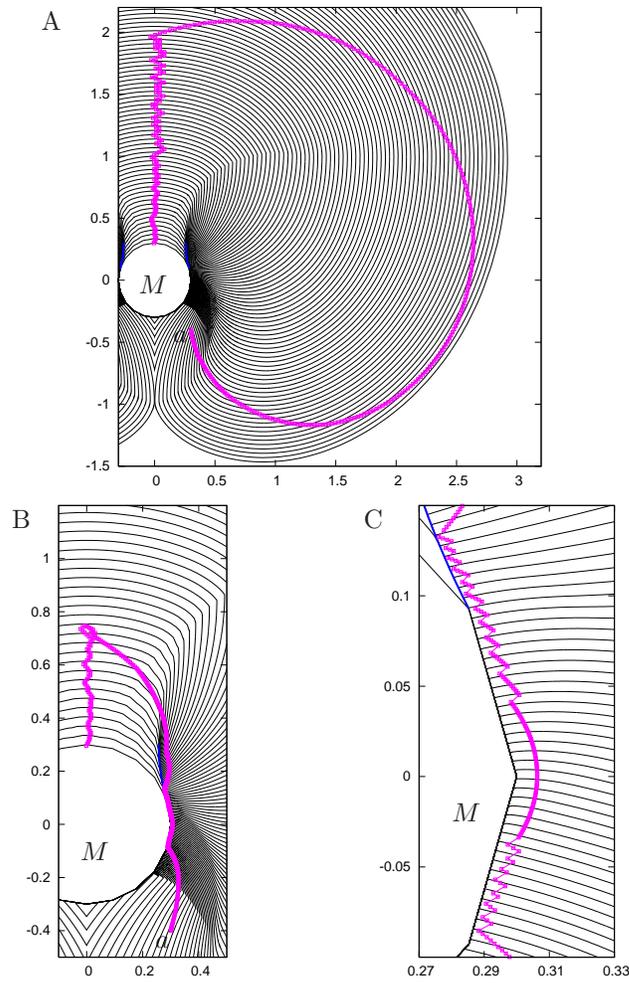
In Figure 39, the trajectories for non-optimal behavior of player  $P$  and optimal behavior of player  $E$  are shown. The control  $u$  of player  $P$  acts in optimal way, whereas the control  $w$  is non-optimal. For Figure 39A,  $w \equiv 1$ . The time of reaching the target set is 7.36. For Figures 39B and C,  $w \equiv -1$



**Figure 37:** Homicidal chauffeur game with more agile pursuer. Simulation results for optimal motions. (A) Initial point  $a = (0.3, -0.4)$ . (B) Initial point  $b = (0.29, 0.1)$ . (C) Enlarged fragment of the trajectory from the point  $b$ .



**Figure 38:** Homicidal chauffeur game with more agile pursuer. Optimal behavior of player  $P$  and random action of player  $E$ . (A) Initial point  $a = (0.3, -0.4)$ . (B) Initial point  $b = (0.29, 0.1)$ .



**Figure 39:** Homicidal chauffeur game with more agile pursuer. Optimal behavior of player  $E$  and non-optimal control  $w$  of player  $P$ . Initial point  $a = (0.3, -0.4)$ . (A)  $w \equiv 1$ . (B)  $w = -0.1$  until the trajectory comes to the vertical axis, after that  $w \equiv 1$ . (C) Enlarged fragment of the trajectory on the left.

until the trajectory comes to the vertical axis, after that  $w \equiv 1$ . Figure 39C demonstrates an enlarged fragment of the trajectory from Fig. 39B. The trajectory goes very close to the terminal set. The reaching time is 5.06.

## 7. Homicidal chauffeur game as a test example

Presently, numerical methods and algorithms for solving antagonistic differential games are intensively developed. Often, the homicidal chauffeur game is used as a test or demonstration example. Some of these papers are [3,25,26,27,29,32].

In the reference coordinates, the game is of the second order in the phase variables. Therefore, one can apply both general algorithms and algorithms taking into account the specifics of the plane. The non-triviality of the dynamics is in that the control  $u$  enters the right hand side of the two-dimensional control system as a factor by the state variables, and that the constraint on the control  $v$  can depend on the phase state. Moreover, the control of player  $P$  can be two-dimensional, as it is in the modification discussed in Section 5, and the target set can be nonconvex like in the problem from Section 3.

Along with the antagonistic statements of the homicidal chauffeur problem, some close but non-antagonistic settings are known as being of great interest for the numerical investigation. In this connection we note the two-target homicidal chauffeur game [12] with players  $P$  and  $E$ , each attempting to drive the state into his target set without being first driven to the target set of his opponent. For the first time, two-target differential games were introduced in [5]. The applied interpretation of such games can be a dogfight between two aircrafts or ships [10,24,28].

## Conclusion

Isaacs' homicidal chauffeur game is important for applications and offers a wide field for mathematical research. A complete solution is obtained by A. Merz in his PhD thesis performed under the supervision of J. Breakwell. The present paper is of a survey character. Both the classic statement of the homicidal chauffeur game and some formulations of the problem with modified objectives of the players or more complicated description of dynamics are considered. Computation results on the construction of level sets of the value function and graphs of the value function are given.

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## Contents

<b>Introduction</b> .....	3
<b>1. Classic statement by R. Isaacs</b> .....	9
<b>2. Investigations by J. Breakwell and A. Merz</b> .....	15
<b>3. Surveillance-evasion game</b> .....	24
<b>4. Acoustic version of the homicidal chauffeur game</b> .....	28
<b>5. Game with a more agile player <math>P</math>.</b> .....	31
<b>6. Optimal strategies</b> .....	36
<b>7. Homicidal chauffeur as a test example</b> .....	39
<b>Conclusion</b> .....	39
<b>Bibliography</b> .....	40

Scientific edition

**Valerii Semenovich Patsko**  
**Varvara Leonidovna Turova**

HOMICIDAL CHAUFFEUR GAME:  
HISTORY AND MODERN STUDIES

Scientific report

Recommended for publication by Scientific Board  
of the Institute of Mathematics and Mechanics  
and NISO UrB RAS

Institute of Mathematics and Mechanics, S.Kovalevskaya str., 16,  
Ekaterinburg, GSP-384, 620219, Russia

Ответственный за выпуск *А. Г. Иванов*

НИСО УрО РАН N 17(09)  
Подписано в печать 15.08.09. Формат 60 × 84/16.  
Бумага типографская. Печать офсетная.  
Усл. печ. л. 2,75. Уч-изд. л. 3,1. Тираж 100. Заказ 153.

Подготовлено в Институте математики и механики УрО РАН.  
620219, Екатеринбург, ГСП-384, ул. С. Ковалевской, 16,  
Институт математики и механики УрО РАН

Отпечатано в типографии  
«Уральский центр академического обслуживания»  
620219, Екатеринбург, ул. Первомайская, 91