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# ON OPTIMALITY OF A DISCONTINUOUS FUNCTION IN A TIME-OPTIMAL DIFFERENTIAL GAME

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Abstract: In this paper, a differential game is considered where the payoff is the time required to reach a given terminal set. Sufficient conditions for coincidence a given discontinuous function (tested function) with the value function of the game are derived. The conditions are formulated in terms of classical notions of $u$- and $v$-stable functions, but additionally fulfillment of so-called correct compressibility condition for closed level sets of the tested function is demanded.
A given example shows that the correct compressibility condition is not excessive.

Keywords: Differential game, Time-optimal control, Value function

## 1. INTRODUCTION

This work is devoted to a problem of deriving conditions on a given discontinuous function to provide its coincidence with the value function of a time-optimal differential game under investigation. The research is carried out in the frame of the positional formalization of differential games introduced in (Krasovskii and Subbotin, 1988).

In the theory of differential games, problems of feedback control under uncertainty and disturbance are investigated. The useful control is considered as an action of the first player, which minimizes a payoff functional on the set of all trajectories of a dynamical system. The disturbance is regarded as a result of control action of the second player, which is aimed to maximize the same functional. The classical approach to solving a differential game consists of finding the value function, which puts an optimal guaranteed result into correspondence with the initial position of the game. On the base of the value function, feedback strategies of the players are constructed.

In case of differentiable value function, the problem of its searching is reduced (Isaacs, 1965) to solving a boundary value problem for a partial differential equation (PDE) of the first order (Isaacs-Bellman equation).

If the value function is non-smooth but continuous, then basic notions for its characterization are continuous $u$-stable and $v$-stable functions (Krasovskii and Subbotin, 1988, p. 145) introduced in the theory of positional differential games. In this case, $u$-stable ($v$-stable) functions under corresponding boundary conditions majorize (minorize) the value function of differential game, which is the only possesses the both properties.

Discontinuous value functions arise, for example, in time-optimal problems. It leads to notions of semicontinuous $u$-stable and $v$-stable functions. With that, the characterization of value functions becomes more complex. Namely, in time-optimal game problem, the value function is the only

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lower semicontinuous $u$-stable function satisfying zero boundary condition on the border of the terminal set, and the function is a pointwise limit of a sequence of upper semicontinuous $v$-stable functions satisfying the same boundary condition. Verification of existence of the sequence as far as its construction are difficult enough even for problems in plane.

In this work, sufficient conditions are suggested for coincidence a discontinuous tested function with the value function of a time-optimal differential game under investigation. The conditions include $u$-stability of lower semicontinuous tested function, $v$-stability of upper closure of the tested function, and fulfilment of correct compressibility condition for the level sets of the tested function.

Properties of $u$- and $v$-stability were researched very well in the theory of differential games. Various infinitesimal criterions of $u$- and $v$-stability of semicontinuous functions were derived (Subbotin, 1995). Let us also remark that the notion of $u$-stable ($v$-stable) function corresponds to the notion of upper (lower) generalized viscosity solution of the first order PDEs (see, for example, (Bardi and Capuzzo-Dolcetta, 1997)).

Checking correct compressibility of a closed set is a self-sufficient problem. Its solving is not obvious in a general case. In this work, the problem of checking correct compressibility of sets is not investigated.

In optimal control theory, the method of dynamical programming is analogues to the approach based on the value function in differential games. If the optimal result function (Bellman function) is differentiable, then the problem of its searching is reduced to solving a corresponding boundary value problem for a PDE of the first order (Bellman equation). In this case, the Bellman function defines an optimal feedback control. If the Bellman function is non-smooth but continuous, then a regular synthesis by Boltyanski (Boltyanski, 1969) can be used to solve the problem in a class of feedback controls. Justification of the regular synthesis is based on the well-known Pontryagin Maximum Principle. If the Bellman function is discontinuous, then construction of an optimal feedback control usually takes into account dynamics peculiarities of each particular problem.

The sufficient conditions suggested in this article are also valid for control problems, which can be considered as special cases of differential game problems (with null constraint on control of the second player). But there are no any simplifications in formulation of the conditions, i.e. checking $v$-stability of the upper closure of a tested function is required despite the second player's absence.

## 2. PROBLEM STATEMENT

A dynamical control problem is considered where the motion of a system is described by the following equation

$$\dot{x}(t) = f(x(t), u(t), v(t)), \quad t \geq 0. \tag{1}$$

Here, $x(t) \in R^n$ is a phase state of the system at an instant $t$; $u(t) \in P$ and $v(t) \in Q$ are controls of the first (minimizing) and the second (maximizing) players; $P$ and $Q$ are compact sets.

Assume that the function $f(x, u, v)$ is continuous in totality of variables, it satisfies the inequality

$$\|f(x, u, v)\| \leq \kappa(1 + \|x\|), \quad \kappa = \text{const} > 0,$$

and the Lipschitzian condition in variable $x$ is fulfilled in any bounded set $\mathcal{X} \subset R^n$, i.e.

$$\|f(x^{(1)}, u, v) - f(x^{(2)}, u, v)\| \leq \lambda(\mathcal{X})\|x^{(1)} - x^{(2)}\|$$

for all $x^{(1)}, x^{(2)} \in \mathcal{X}$, $u \in P$, $v \in Q$. In addition, let us suppose that the following saddle-point condition is satisfied for all $x, p \in R^n$:

$$\min_{u \in P} \max_{v \in Q} \langle p, f(x, u, v)\rangle = \max_{v \in Q} \min_{u \in P} \langle p, f(x, u, v)\rangle.$$

The aim of the first player is to approach the phase point $x(t)$ from an initial position $x_0$ to a given closed set $M \subset R^n$. The second player tries either to prevent an encounter with $M$ or to maximize the time till it occurs.

A positional strategy $U$ for the first player is an arbitrary function $U : R^n \to P$. The strategy $U$ generates a bundle $X_1(x_0, U)$ of constructive motions. A constructive motion $x(\cdot)$ is defined as a function, which is, in any bounded interval $(0, \bar{\vartheta})$, a uniform limit of a sequence of trajectories $\{x^{(k)}(\cdot)\}_1^\infty$, such that $x^{(k)}(0) = x_0$ and for $t \in [\tau_i^{(k)}, \tau_{i+1}^{(k)})$, $i = 1, 2, \ldots$, the equation

$$\dot{x}^{(k)}(t) = f(x^{(k)}(t), U(x^{(k)}(\tau_i^{(k)})), v^{(k)}(t))$$

is hold. Here, $v^{(k)}(\cdot) : [0, \infty) \to Q$ is a measurable function, the sequence $\{\tau_i^{(k)}\}_{i=0}^\infty$ is such that

$$0 = \tau_0^{(k)} < \tau_1^{(k)} < \ldots < \tau_i^{(k)} < \tau_{i+1}^{(k)} < \ldots,$$

and $\sup_i(\tau_{i+1}^{(k)} - \tau_i^{(k)}) \to 0$ as $k \to \infty$.

A positional strategy $V : R^n \to Q$ for the second player and the bundle $X_2(x_0, V)$ of constructive motions generated by $V$ are defined similarly.

Let $J(x(\cdot)) = \min\{t \geq 0 : x(t) \in M\}$. Under the conditions assumed for the function $f(x, u, v)$, the value

$$T(x_0; M) = \min_U \sup\{J(x(\cdot)) : x(\cdot) \in X_1(x_0, U)\}$$
$$= \sup_V \inf\{J(x(\cdot)) : x(\cdot) \in X_2(x_0, V)\}$$

of the game exists for any $x_0 \in R^n$. The function $T(\cdot\,; M) : R^n \to [0, \infty]$ is known as the *value function* of the time-optimal differential game.

The purpose of the work consists of finding such conditions on a function $\varphi(\,\cdot\,) : \Omega \to [0, \infty]$, that the equation $\varphi(x) = T(x; M)$ is valid for $x \in \Omega$. Here, $\Omega \subseteq R^n$ is a closed set and $M \subset \Omega$. Desirable conditions should use the properties of the function $\varphi(\,\cdot\,)$ only and should not require any additional constructing.

## 3. PROPERTIES OF THE VALUE FUNCTION

Let us give main properties of the value function to use later on.

Let us introduce level sets of the value function:

$$W(t; M) = \{x \in R^n : T(x; M) \le t\}, \quad t \ge 0.$$

For any $\tau > 0$ and $x \notin W(\tau; M)$, the equation

$$T(x; M) = T(x; W(\tau; M)) + \tau$$

is valid. The fact is easily seen taking into account the definition of the value function.

From the results of (Krasovskii and Subbotin, 1988; Subbotin, 1995), it follows that $T(\,\cdot\,; M)$ is a lower semicontinuous function, $M = W(0; M)$, and the $u$-stability property is fulfilled:

($\mathrm{T}_u$) for any $y_0 \in R^n \setminus M$ and $v_* \in Q$, there exists $\tau > 0$ and such a solution $y(\cdot) : [0, \tau] \to R^n$ of the differential inclusion

$$\dot{y}(t) \in \mathrm{co}\,\{f(y(t), u, v_*), u \in P\}, \quad y(0) = y_0,$$

that either the inequality

$$T(y(t); M) \le T(y_0; M) - t$$

holds for all $t \in [0, \tau]$, or $x(t) \in M$ for a certain $t \in [0, \tau]$.

For the upper closure

$$T^*(x; M) = \limsup_{y \to x} T(x; M)$$

of the value function the following $v$-stability property is fulfilled (Subbotin, 1995):

($\mathrm{T}_v$) for any $y_0 \in R^n \setminus M$ and $u_* \in P$, there exists $\tau > 0$ and such a solution $y(\cdot) : [0, \tau] \to R^n$ of the differential inclusion

$$\dot{y}(t) \in \mathrm{co}\,\{f(y(t), u_*, v), v \in Q\}, \quad y(0) = y_0,$$

that the inequality

$$T^*(y(t); M) \ge T^*(y_0; M) - t$$

holds for all $t \in [0, \tau]$.

## 4. CORRECTLY COMPRESSIBLE SETS

Let $\mathcal{D} \subset R^n$ be a closed set and $\mathrm{int}\,\mathcal{D}$ denotes interior of $\mathcal{D}$. Under the condition $\mathrm{int}\,\mathcal{D} \ne \emptyset$, define

$$\mathcal{D}^{[\varepsilon]} = \{x \in \mathcal{D} : \mathbf{B}(x, \varepsilon) \subseteq \mathcal{D}\}, \quad \varepsilon > 0,$$

$$\varepsilon_{\mathcal{D}} = \max\{\varepsilon > 0 : \mathcal{D}^{[\varepsilon]} \ne \emptyset\}.$$

Here, $\mathbf{B}(x, \varepsilon)$ is a ball of radius $\varepsilon$ with the center in $x$.

The following assertion will be useful.

*Lemma 1.* Let $\mathcal{D}$ be a closed set, $\mathrm{int}\,\mathcal{D} \ne \emptyset$, $x_* \in R^n$, and

$$\lim_{\varepsilon \to +0} T(x_*; \mathcal{D}^{[\varepsilon]}) = T(x_*; \mathcal{D}).$$

Then the function $T(\cdot; \mathcal{D})$ is continuous at $x_*$.

The proof of Lemma 1 is omitted.

*Definition.* A set $\mathcal{D} \subset R^n$ is called *correctly compressible* with respect to the dynamic (1) if there exists $\vartheta > 0$, such that

(C1) $W(\vartheta; \mathcal{D}) \ne \mathcal{D}$ and $W(t; \mathcal{D}) = \overline{\mathrm{int}\,W(t; \mathcal{D})}$ for any $t \in [0, \vartheta]$;

(C2) for any $x \in \mathrm{int}\,W(\vartheta; \mathcal{D}) \setminus \mathcal{D}$, it holds

$$\lim_{\varepsilon \to +0} T(x; \mathcal{D}^{[\varepsilon]}) = T(x; \mathcal{D}).$$

Observe that if $W(\vartheta; \mathcal{D}) \ne \mathcal{D}$, then the condition ($\mathrm{T}_u$) of $u$-stability implies the function $T(\cdot; \mathcal{D})$ takes all values from the interval $(0, \vartheta)$. This allows to choose the value $\vartheta > 0$, satisfying conditions (C1) and (C2), as near to zero as desired.

Let us give simple conditions, which provide the property of correct compressibility of a set $\mathcal{D}$.

Let the set $\mathcal{D}$ has a smooth boundary $\partial\mathcal{D}$, $\mathcal{D} = \overline{\mathrm{int}\,\mathcal{D}}$, and for any point $x \in \partial\mathcal{D}$ the equation

$$\min_{u \in P} \max_{v \in Q} \langle \nu(x), f(x, u, v) \rangle < 0 \tag{2}$$

is fulfilled, where $\nu(x)$ is an exterior normal to the set $\mathcal{D}$ at point $x \in \partial\mathcal{D}$.

Choose $\vartheta > 0$ and $x_* \in W(\vartheta; \mathcal{D})$. Condition (2) and the definition of the value function imply that for any $\tau > 0$ the first player guarantees approaching the set $\mathrm{int}\,\mathcal{D}$ from the point $x_*$ within the time interval $[0, T(x_*; \mathcal{D}) + \tau]$. It means that there exists $\varepsilon_* > 0$, such that

$$T(x_*; \mathcal{D}^{[\varepsilon_*]}) \le T(x_*; \mathcal{D}) + \tau.$$

From here, taking into account lack of increase of the quantities $T(x_*; \mathcal{D}^{[\varepsilon]})$ as $\varepsilon \to +0$, we have the limit relation

$$\lim_{\varepsilon \to +0} T(x_*; \mathcal{D}^{[\varepsilon]}) = T(x_*; \mathcal{D}).$$

So, condition (C2) from the definition of correctly compressible set is fulfilled for any $\vartheta > 0$.

For any $x_* \in R^n \setminus \cup_{\vartheta \ge 0} W(\vartheta; \mathcal{D})$ we get

$$T(x_*; \mathcal{D}) = T(x_*; \mathcal{D}^{[\varepsilon]}) = \infty, \quad \varepsilon \in (0, \varepsilon_{\mathcal{D}}].$$

Due to Lemma 1, the value function $T(\cdot; \mathcal{D})$ is continuous on $R^n$. Let us remark that conditions of continuity of the value function under weaker assumptions on the set $\mathcal{D}$ was proved in (Bardi and Capuzzo-Dolcetta, 1997).

The validity of condition (C1) for any $\vartheta > 0$ follows from continuity of the function $T(\cdot;\mathcal{D})$ and the property $(\mathrm{T}_u)$.

Thus, the set $\mathcal{D}$ is correctly compressible.

More complicated sufficient conditions for correct compressibility are connected with a discontinuous value function and not investigated here.

## 5. THEOREM ABOUT SUFFICIENT CONDITIONS

*Theorem.* Let $\Omega \subseteq R^n$ and $M \subset \Omega$ be closed sets, a function $\varphi(\,\cdot\,) : \Omega \to [0,\infty]$ is lower semicontinuous, and the following conditions are fulfilled:

(A1) $\varphi(x) = 0,\ x \in M$;

(A2) ($u$-stability) for any $y_0 \in \Omega \setminus M$ and $v_* \in Q$, there exists $\tau > 0$ and such a solution $y(\cdot) : [0,\tau] \to \Omega$ of differential inclusion

$$\dot{y}(t) \in \mathrm{co}\,\{f(y(t),u,v_*), u \in P\}, \quad y(0) = y_0,$$

that either the inequality

$$\varphi(y(t)) \le \varphi(y_0) - t, \quad t \in [0,\tau],$$

is valid, or $y(t) \in M$ for a certain $t \in [0,\tau]$;

(A3) ($v$-stability) for any $y_0 \in \Omega \setminus M$ and $u_* \in P$, there exists $\tau > 0$ and such a solution $y(\cdot) : [0,\tau] \to R^n$ of differential inclusion

$$\dot{y}(t) \in \mathrm{co}\,\{f(y(t),u_*,v), v \in Q\}, \quad y(0) = y_0,$$

that the inequality

$$\varphi^*(y(t)) \ge \varphi^*(y_0) - t, \quad t \in [0,\tau],$$

is valid, where

$$\varphi^*(x) = \begin{cases} \limsup\limits_{z\to x} \varphi(z), & \text{if } x \in \mathrm{int}\,\Omega, \\ \sup\limits_{z\in\Omega} \varphi(z), & \text{if } x \notin \mathrm{int}\,\Omega; \end{cases} \qquad (3)$$

(A4) the level sets

$$D(t) = \{x \in \Omega : \varphi(x) \le t\}, \quad 0 < t < \sup_{z\in\Omega} \varphi(z),$$

are correctly compressible.

Then $\varphi(x) = T(x;M)$, $x \in \Omega$.

The upper semicontinuous function $\varphi^*(\cdot) : R^n \to [0,\infty]$ defined by (3) will be called *upper closure* of the function $\varphi(\cdot) : \Omega \to [0,\infty]$.

*Remark* 1. Let conditions (A1)–(A3) are fulfilled for the function $\varphi(\cdot)$ and the property of correct compressibility of the level sets $D(t)$ defaults only in a single point $a \in (0, \sup_{z\in\Omega} \varphi(z))$. In this case, the theorem about sufficient condition can firstly be applied to the function $\varphi(\cdot) : D(a) \to [0,\infty)$. It gives the equality $\varphi(x) = T(x;M)$, $x \in D(a)$. Secondly, introducing a notation

$$M_1 = D(a), \quad \varphi_1(x) = \begin{cases} \varphi(x) - a, & \text{if } x \notin M_1, \\ 0, & \text{if } x \in M_1, \end{cases}$$

one can apply the theorem to the function $\varphi_1(\cdot) : \Omega \to [0,\infty]$ and a time-optimal differential game with the terminal set $M_1$. From here, using the relation $T(x;M_1) = T(x;M) - a$, one gets the equality $\varphi(x) = T(x;M)$ for all $x \in \Omega$.

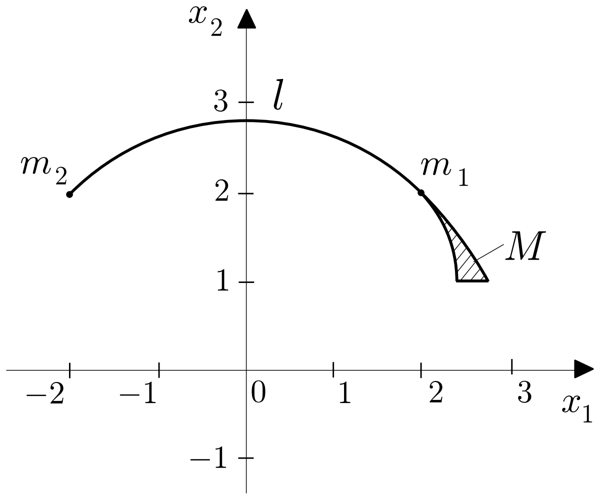


Fig. 1. Example: the terminal set $M$ and the set of solvability $l$.

*Remark* 2. Properties $(\mathrm{T}_u)$ and $(\mathrm{T}_v)$ show that conditions (A1)–(A3) are necessary for the value function. If the function $\varphi(\cdot)$ is continuous, then conditions (A1)–(A3) are necessary and sufficient (Subbotin, 1995) for the equality $\varphi(x) = T(x;M)$, $x \in \Omega$.

Let us give an example to show that in case of discontinuous function $\varphi(\cdot)$ one cannot refuse condition (A4).

Consider a linear dynamical system

$$\dot{x}_1 = x_2 + v_1, \quad \dot{x}_2 = -x_1 + u + v_2,$$
$$x \in R^2, \quad |u| \le 1, \quad v \in Q,$$

where

$$Q = \{v \in R^2 : |v_1| + |v_2| \le 1\}.$$

The set $M$ is defined by the following system of inequalities (Fig. 1):

$$(x_1 - 1)^2 + (x_2 - 1)^2 \ge (\sqrt{2})^2, \quad x_1 \ge 2,$$
$$(x_1 + 1)^2 + (x_2 + 1)^2 \le (3\sqrt{2})^2, \quad x_2 \ge 1.$$

Set $m_1 = (2,2)^T$ and $m_2 = (-2,2)^T$. Let us denote by $l$ the smaller arc of a circle by radius $2\sqrt{2}$ with the center in the origin, joining the points $m_1$ and $m_2$ (Fig. 1). It can be proved that the value function $T(\cdot;M)$ is finite on the set $l \cup M$ only. The proof of the fact is omitted here.

Define the value $\varphi(x) = 2T(x;M)$, $x \in R^2$. The property of $u$-stability of the value function $T(\cdot;M)$ implies that the function $\varphi(\cdot)$ also possesses the $u$-stability property. Since $\varphi^*(x) = T^*(x;M)$, $x \in R^2$, the function $\varphi^*(\cdot)$ have the $v$-stability property. Here, $T^*(\cdot;M)$ and $\varphi^*(\cdot)$ are upper closures of the functions $T(\cdot;M)$ and $\varphi(\cdot)$ defined by (3), where $\Omega = R^2$.

Thus, the function $\varphi(\cdot) : R^2 \to [0,\infty]$ satisfies conditions (A1)–(A3), but $\varphi(x) \neq T(x;M)$, $x \in l \setminus \{m_1\}$. Condition (A4) is not fulfilled for $\varphi(\cdot)$.

## 6. PROOF OF THE THEOREM

The following lemmas will be used in the proof.

*Lemma 2.* Let a closed set $M \subset R^n$ and a lower semicontinuous function $\varphi(\,\cdot\,) : \Omega \to [0,\infty]$ satisfy conditions (A1), (A2). Then

$$T(x_0;M) \leq \varphi(x_0), \quad x_0 \in \Omega.$$

*Proof.* If $x_0 \in M$, then condition (A1) gives $\varphi(x_0) = T(x_0;M) = 0$. If $x_0 \in \Omega \setminus M$ and $\varphi(x_0) = \infty$, then $T(x_0;M) \leq \infty = \varphi(x_0)$.

Choose $x_0 \in \Omega \setminus M$ and set $\vartheta_* = \varphi(x_0) < \infty$, $\vartheta_0 = T(x_0;M)$. Let us prove that $\vartheta_0 \leq \vartheta_*$.

Define

$$W_* = \{(t,x) \in [0,\vartheta_*] \times \overline{R^n \setminus M} : \varphi(x) \leq \vartheta_* - t\}.$$

Condition (A2) implies that $W_*$ is a $u$-stable bridge (Krasovskii and Subbotin, 1988) in a problem of approaching a point $(t, x(t))$ with the set $[0,\vartheta_*] \times M$. Since $(0,x_0) \in W_*$, the first player has (Krasovskii and Subbotin, 1988) a positional strategy, which guarantees approaching the set $M$ in the time interval $[0,\vartheta_*]$.

The definition of the value function gives that for any $\delta > 0$ the second player has a positional strategy, which guarantees avoidance a neighbourhood of the set $M$ in the time interval $[0,\vartheta_0 - \delta]$. This implies the inequality $\vartheta_0 - \delta < \vartheta_*$ for any $\delta > 0$. Thus, $\vartheta_0 \leq \vartheta_*$, which completes the proof.

*Lemma 3.* Let a closed set $M \subset R^n$ and a lower semicontinuous function $\varphi(\,\cdot\,) : \Omega \to [0,\infty]$ satisfy conditions (A1), (A3). Besides, let $\operatorname{int} M \neq \emptyset$. Then $\varphi^*(x_0) \leq T(x_0;M^{[\varepsilon]})$ for any $x_0 \in R^n$ and $\varepsilon \in (0,\varepsilon_M]$.

*Proof.* Let $x_0 \in R^n$ and $\vartheta^* = \varphi^*(x_0) < \infty$. Define

$$W^* = \{(t,x) \in [0,\vartheta^*] \times R^n : \varphi^*(x) \geq \vartheta^* - t\}.$$

Condition (A3) implies that $W^*$ is $v$-stable set (Krasovskii and Subbotin, 1988). As $(0,x_0) \in W^*$, the second player has (Krasovskii and Subbotin, 1988) a positional strategy, which guarantees keeping the system in the set $W^*$ in the time interval $[0,\vartheta^*]$. Since

$$([0,\vartheta^*] \times M^{[\varepsilon]}) \cap W^* = \emptyset, \quad \varepsilon \in (0,\varepsilon_M],$$

the second player avoids a neighbourhood of the set $M^{[\varepsilon]}$ in the time interval $[0,\vartheta^*]$. But the first player possesses a positional strategy, which leads the system to the set $M^{[\varepsilon]}$ in the time interval $[0, T(x_0;M^{[\varepsilon]})]$. Thus, $\vartheta^* < T(x_0;M^{[\varepsilon]})$ for any $\varepsilon \in (0,\varepsilon_M]$.

Let $x_0 \in R^n$ and $\varphi^*(x_0) = \infty$. Define

$$W_\infty = [0,\infty) \times \{x \in R^n : \varphi^*(x) = \infty\}.$$

The set $W_\infty$ is closed and $(0,x_0) \in W_\infty$. Condition (A3) implies that $W_\infty$ is a $v$-stable set. Since $\varphi^*(x) = 0$ for any $x \in \operatorname{int} M$, we have

$$([0,\infty) \times \operatorname{int} M) \cap W_\infty = \emptyset.$$

Then $([0,\infty) \times M^{[\varepsilon]}) \cap W_\infty = \emptyset$ for any $\varepsilon \in (0,\varepsilon_M]$. Consequently, for any $\vartheta > 0$ the second player has a positional strategy, which guarantees avoidance a neighbourhood of the set $M^{[\varepsilon]}$ in the time interval $[0,\vartheta]$. So, $T(x_0;M^{[\varepsilon]}) = \infty$ for all $\varepsilon \in (0,\varepsilon_M]$. The lemma is proved.

*Lemma 4.* Let closed sets $\mathcal{D}_\tau \subset R^n$, $\tau > 0$, be decreasing by inclusion as $\tau \to +0$ and $\cap_{\tau>0}\mathcal{D}_\tau = M$. Then

$$\lim_{\tau \to +0} T(x;\mathcal{D}_\tau) = T(x;M), \quad x \in R^n. \tag{4}$$

The proof of Lemma 4 is omitted.

*Proof of the Theorem.* The conclusion of the Theorem is obviously true if $M = \Omega$.

Let $M \neq \Omega$. Condition (A1) gives the relation $W(0;M) = M \subseteq D(0)$. Condition (A2) of $u$-stability implies absence of local minimum points of the function $\varphi(\cdot)$, where it takes finite values, outside of the set $M$. Thus, the equality $W(0;M) = D(0)$ is valid.

Since $M \neq \Omega$ and $D(0) = M$, we have $\sup_{x\in\Omega} \varphi(x) > 0$.

Choose $\tau \in (0, \sup_{x\in\Omega} \varphi(x))$. Define the function $\varphi_\tau(\cdot) : \Omega \to [0,\infty]$ as follows

$$\varphi_\tau(x) = \begin{cases} \varphi(x) - \tau, & \text{if } x \notin D(\tau), \\ 0, & \text{if } x \in D(\tau). \end{cases}$$

Let us show that

$$T(x;D(\tau)) = \varphi_\tau(x), \quad x \in \Omega. \tag{5}$$

For brevity let $D_\tau = D(\tau)$. The set $D_\tau$ and the lower semicontinuous function $\varphi_\tau(\,\cdot\,) : \Omega \to [0,\infty]$

satisfy conditions (A1)–(A4), where the notations $M$ and $\varphi(\,\cdot\,)$ are replaced by $D_\tau$ and $\varphi_\tau(\cdot)$.

Define

$$\Theta = \sup_{x\in\Omega} \varphi_\tau(x), \quad E(t) = \{x \in \Omega : \varphi_\tau(x) \le t\},$$

$$\gamma = \sup\{\vartheta \in [0, \Theta) \colon W(t; D_\tau) = E(t) \; \forall\, t \in [0, \vartheta]\}.$$

Let us remark, that for any $\vartheta \in [0, \gamma)$ and $x \in E(\vartheta)$ the equality $\varphi_\tau(x) = T(x; D_\tau)$ is valid. In fact, set $t = T(x; D_\tau)$. Since $E(\vartheta) = W(\vartheta; D_\tau)$, we have $t \in [0, \vartheta]$ and $x \in W(t; D_\tau) = E(t)$. Due to Lemma 2, the equation $t \le \varphi_\tau(x)$ is fulfilled. On the other hand, the relation $x \in E(t)$ implies $\varphi_\tau(x) \le t$. Thus, $\varphi_\tau(x) = T(x; D_\tau)$.

Consider the following cases.

*Case* 1: $\gamma = \infty$. For any $\vartheta \ge 0$ and $x \in E(\vartheta)$ we have $\varphi_\tau(x) = T(x; D_\tau)$. If $x \in \Omega \setminus \cup_{\vartheta\ge0} E(\vartheta)$, then taking into account the definition of $\gamma$ we get $x \notin \cup_{\vartheta\ge0} W(\vartheta; D_\tau)$, and, therefore, $T(x; D_\tau) = \infty$. By Lemma 2, the equality $\varphi_\tau(x) = \infty$ is valid.

*Case* 2: $\gamma < \infty$ and $\gamma = \Theta$. If $x \in E(\vartheta)$ for a certain $\vartheta \in [0, \gamma)$, then the equation $\varphi_\tau(x) = T(x; D_\tau)$ is valid.

Let $x \in \Omega \setminus \bigcup_{\vartheta\in[0,\gamma)} E(\vartheta)$. Taking into account Lemma 2, we have $\varphi_\tau(x) \ge T(x; D_\tau) \ge \gamma = \Theta$. By definition of the value $\Theta$, we get the equation $T(x; D_\tau) = \gamma$. The property $(\mathrm{T}_u)$ of $u$-stability of the value function implies the existence of such a sequence $\{x_k\}_1^\infty$, that $t_k = T(x_k; D_\tau) < \gamma$ and $x_k \to x$ as $k \to \infty$. Since $W(t_k; D_\tau) = E(t_k)$ and $x_k \in E(t_k)$, the equation $\varphi_\tau(x_k) = T(x_k; D_\tau)$ is true. Lemma 2 and lower semicontinuity of the function $\varphi_\tau(\cdot)$ give

$$\gamma = T(x; D_\tau) \le \varphi_\tau(x) \le \limsup_{k\to\infty} \varphi_\tau(x_k)$$

$$= \limsup_{k\to\infty} T(x_k; D_\tau) = \limsup_{k\to\infty} t_k \le \gamma.$$

Thus, the equality $\varphi_\tau(x) = T(x; D_\tau)$ is valid for all $x \in \Omega$.

*Case* 3: $\gamma \in [0, \Theta)$. Let us introduce a notation:

$$\mathcal{D} = E(\gamma), \quad \widetilde{\varphi}(x) = \begin{cases} \varphi_\tau(x) - \gamma, & \text{if } \; x \notin \mathcal{D}, \\ 0, & \text{if } \; x \in \mathcal{D}, \end{cases}$$

$$\mathbf{D}(t) = \{x \in \Omega : \widetilde{\varphi}(x) \le t\}.$$

Observe that $\sup_{x\in\Omega} \widetilde{\varphi}(x) = \Theta - \gamma$.

The set $\mathcal{D}$ and the lower semicontinuous function $\widetilde{\varphi}(\,\cdot\,) : \Omega \to [0, \infty]$ satisfy conditions (A1)–(A3), where the notations $M$ and $\varphi(\,\cdot\,)$ are replaced by $\mathcal{D}$ and $\widetilde{\varphi}(\,\cdot\,)$. Taking into account condition (A4) of the Theorem and the equality $\mathcal{D} = D(\tau + \gamma)$, where $0 < \tau + \gamma < \sup_{x\in\Omega} \varphi(x)$, we find that the set $\mathcal{D}$ is correctly compressible. Choose the value $\vartheta > 0$, satisfying conditions (C1) and (C2), in a way that $\vartheta < \Theta - \gamma$.

Let us show that

$$W(t; \mathcal{D}) = \mathbf{D}(t), \quad t \in (0, \vartheta). \tag{6}$$

Choose $t \in (0, \vartheta)$. Taking into account Lemma 2, we have

$$\mathbf{D}(t) \subseteq W(t; \mathcal{D}) = \overline{\operatorname{int} W(t; \mathcal{D})}.$$

Assume that $W(t; \mathcal{D}) \setminus \mathbf{D}(t) \ne \emptyset$. Since $\widetilde{\varphi}(\,\cdot\,)$ is a lower semicontinuous function, $\mathbf{D}(t)$ is a closed set. Thus, there exists $x \in \operatorname{int} W(t; \mathcal{D}) \setminus \mathbf{D}(t)$. We have $T(x; \mathcal{D}) \in (0, t]$.

If $x \in \Omega$, then the relation $x \notin \mathbf{D}(t)$ implies $t < \widetilde{\varphi}(x)$. So, $t < \widetilde{\varphi}^*(x)$, where $\widetilde{\varphi}^*(\cdot)$ is the upper closure of the function $\widetilde{\varphi}(\cdot)$. If $x \notin \Omega$, then $\widetilde{\varphi}^*(x) = \Theta - \gamma > t$. Hence,

$$T(x; \mathcal{D}) \le t < \widetilde{\varphi}^*(x). \tag{7}$$

On the other hand, Lemma 3 implies the inequality

$$\widetilde{\varphi}^*(x) \le T(x; \mathcal{D}^{[\varepsilon]}), \quad \varepsilon \in (0, \varepsilon_{\mathcal{D}}].$$

From condition (C2) of correct compressibility of the set $\mathcal{D}$, it follows that

$$T(x; \mathcal{D}) = \lim_{\varepsilon\to+0} T(x; \mathcal{D}^{[\varepsilon]}).$$

Therefore, $\widetilde{\varphi}^*(x) \le T(x; \mathcal{D})$, that contradicts (7). Hence, the property (6) is proved.

Taking into account that

$$W(t; \mathcal{D}) = W(\gamma + t; D_\tau), \quad \mathbf{D}(t) = E(\gamma + t)$$

for any $t \in (0, \vartheta)$, we get $W(t; D_\tau) = E(t)$ for any $t \in [0, \gamma + \vartheta)$. This contradicts the definition of $\gamma$. Consequently, Case 3 is impossible.

Thus, the relation (5) is proved.

Choose $x_* \in \Omega \setminus M$. For any rather small $\tau > 0$ we have $\varphi_\tau(x_*) = \varphi(x_*) - \tau$. Due to Lemma 4 and the equality (5), we get

$$T(x_*; M) = \lim_{\tau\to+0} T(x_*; D(\tau)) = \varphi(x_*).$$

The last relation completes the proof.

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