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CONTROL IN STOCHASTIC SYSTEMS AND UNDER UNCERTAIN CONDITIONS

Information Sets in the Problem of Observation of Aircraft Motion in a Horizontal Plane

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Abstract—The estimation from above of information sets characterizing the phase states of an aircraft moving in a horizontal plane is considered. It is based on the method of roughening from above of the attainability sets of the nonlinear control system under investigation. The information sets are constructed in the three-dimensional phase space on the basis of measurements of the two-dimensional geometric position taking into account some known constraints on the measurement errors. The algorithms proposed allow us to perform computations in the real-time mode. Some simulation results are presented.

0. INTRODUCTION

In contemporary observation and control theory, for describing the state of a dynamic system under conditions of inexact measurements, both the probabilistic approach [1] and a deterministic method based on constructing information sets [2–7] are used.

By an information set, we mean the totality of all phase states of a system that are consistent with the obtained measurements. An information set can be treated as a *generalized* state of the system.

In the paper, we consider, in the model statement, the observation problem for an aircraft moving in a horizontal plane. The motion dynamics is described [8–12] by the system of differential equations

$$\dot{x} = V\cos\varphi, \quad \dot{y} = V\sin\varphi, \quad \dot{\varphi} = ku/V; \qquad (0.1)$$

$$V = \text{const} > 0, \quad k = \text{const} > 0, \quad |u| \le 1.$$

Here, x and y are the geometric coordinates, φ is the directing angle of the velocity vector, V is the magnitude of the velocity, and u is an unknown control action. Angle φ is measured counterclockwise from the horizontal axis (Fig. 1).

The current information about the motion of the aircraft is supplied in the form of results of measurements (readings) of its position in the x, y plane. Some geometrical constraints on the reading errors are known. A reading received at an instant t_* is associated with the uncertainty set $H(t_*)$, which is the totality of all phase states (x, y, φ) consistent with the received reading and with the specified constraints on its error. We assume that set $H(t_*)$ is cylindrical in the φ coordinate and has a convex projection onto the x, y plane. The totality of all phase states at instant t consistent with the uncertainty sets accumulated up to this instant composes the information set $I(t)$.

In a discrete observation scheme on a grid of instants t_j, the information set $I(t_j)$ is obtained as the intersection of the prediction set $G(t_j)$ with the uncertainty set $H(t_j)$. The set $G(t_j)$ is the attainability set [2–5, 7, 12] of system (0.1) at instant t_j constructed from set $I(t_{j-1})$ taken as the initial set for system (0.1) at the previous observation instant t_{j-1}. The set $H(t_j)$ corresponds to the observation at instant t_j.

The information sets in the problem considered are nonconvex, since the attainability sets are not convex. In the paper, a version of roughing from above the prediction sets (hence, the information sets) is proposed. In the numerical procedure, extended prediction set $\mathbf{G}(t_j)$ and extended information set $\mathbf{I}(t_j)$ are represented by a grid of nodes in the φ coordinate and by a set of sections $\{\mathbf{G}_\varphi(t_j)\}$ (respectively, $\{\mathbf{I}_\varphi(t_j)\}$) in the form of convex

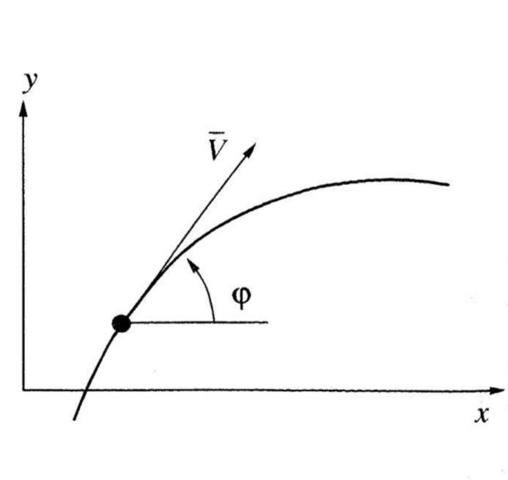

Fig. 1. Coordinate system.

polyhedrons in the x, y plane. The basic operations are those of constructing the convex hull of a union of convex polyhedrons and of intersecting convex polyhedrons. The developed algorithm allows one to construct the information sets in the real-time mode.

Extended prediction set $\mathbf{G}(t_j)$ gives an upper estimate for the attainability set $G(t_j)$ of control system (0.1) with the initial set $\mathbf{I}(t_*)$, where $t_* < t_j$ is the instant of the last reading. To study the character of the arising error, we compare three-dimensional sets $G(t)$ and $\mathbf{G}(t)$ and analyze their projections onto the plane of x, y coordinates. As the initial set, we take a point in the three-dimensional space. The algorithm for constructing the three-dimensional attainability set is based on a proposition (formulated in the paper) on the number of control switches steering toward the boundary of the attainability set.

In the final section of the paper, results of constructing the information sets for various versions of the geometry of uncertainty sets and for different numbers of vertices of convex polyhedrons approximating the sections of the information set are presented.

In this paper, some results of studies [13, 14] mainly devoted to the four-dimensional version of system dynamics, where velocity V is not assumed constant, are used. The three-dimensional variant is simpler, and the proposed ideas of constructing the information sets become more transparent in this case.

1. STATEMENT OF THE PROBLEM ON CONSTRUCTING THE INFORMATION SETS

We assume that the motion dynamics satisfies system (0.1). The values of the φ coordinate are either considered on the infinite axis $(-\infty, \infty)$ or calculated modulo 2π. In the latter case, the values of φ differing by a multiple of 2π are identified. Below, we write formulas for the first case of variation of the φ coordinate. In the second case, when we refer to the calculations of the values of φ, we always mean (mod 2π).

At discrete instants, some results of measurements of the aircraft location on the x, y plane arrive. Each reading is associated with an *uncertainty set* (US) that is the totality of all states (x, y, φ) consistent with this reading under some known constraints on the reading error. For example, if a reading $(\hat{x}, \hat{y})$ arrives at an instant and the maximal radial error of the reading is σ, then the geometric state (unknown to us) at this instant is located in the circle of radius σ with the center at the point $(\hat{x}, \hat{y})$. The uncertainty set of such a reading is the cylinder in the three-dimensional space whose projection onto the x, y plane coincides with this circle.

We assume that uncertainty set $H(t)$ of any current reading is cylindrical in the φ coordinate and is completely determined by its projection $H^{\#}(t)$ onto the x, y plane

$$H(t) = H^{\#}t \times \{\varphi\}. \tag{1.1}$$

In what follows, we assume that sets $H^{\#}(t)$ are convex.

By an *information set* (IS) $I(t)$, we mean the totality of all states (x, y, φ) of system (0.1) at instant t which are consistent with the uncertainty sets obtained up to instant t.

It is required to develop an algorithm for constructing information sets.

2. SCHEME OF CONSTRUCTING INFORMATION SETS

2.1. Formal Description of Information Sets

We assume that an initial information set $I(t_0)$ is known. It is formed on the basis of some preliminary information and by the uncertainty set of the initial reading.

Suppose that, at an instant t_*, the information set $I(t_*)$ is constructed and that the next reading is expected at instant $t^* > t_*$. We define the *prediction set* $G(t^*)$ as the attainability set of system (0.1) at instant t^* from the states that belong to set $I(t_*)$ at instant t_*:

$$G(t^*) = \bigcup_{u(\cdot),\, z \in I(t_*)} \xi(t^*; t_*, z, u(\cdot)).$$

Here, $\xi(t^*; t_*, z, u(\cdot))$ is the solution to the system of differential equations (0.1) determined up to instant t^* for an initial phase state z at instant t_* and for a piecewise continuous control $u(\cdot)$ satisfying the condition $|u(t)| \le 1$ for any t.

The uncertainty set contains some new information about the system; therefore, set $I(t^*)$ is defined as the intersection of the prediction set $G(t^*)$ with the uncertainty set $H(t^*)$ of the reading that arrived:

$$I(t^*) = G(t^*) \cap H(t^*). \tag{2.1}$$

If, at instant t^*, the reading is absent, then the intersection operation is not executed and it is assumed that the current information set $I(t^*)$ coincides with the current prediction set $G(t^*)$. Formally, we can assume that the uncertainty set of the absent reading coincides with the entire space $\{x, y, \varphi\}$.

Thus, at any current instant, the information set is determined by the initial set $I(t_0)$ and the uncertainty sets of the readings that arrived up to this instant.

System (0.1) is nonlinear, and the prediction set is nonconvex and has a complex structure. Hence, intersection (2.1) has a rather complex structure. To efficiently describe the information set, it is necessary to use some simplifications. We will do this by making use of the specific features of the system in a reasonable way.

2.2. Equivalent Representation of Information Sets

We rewrite expression (2.1), defining the information set in an equivalent form that is more convenient for us.

Consider projection $I^{\diamond}(t_*)$ of set $I(t_*)$ onto the φ axis. We associate any point $\varphi \in I^{\diamond}(t_*)$ with section $I_\varphi(t_*)$ of the information set $I(t_*)$ by the plane $\varphi = \text{const}$. The projections of these sections onto the x, y plane will be considered:

$$I_\varphi(t_*) = \{(x, y): (x, y, \varphi) \in I(t_*)\}.$$

We represent the information set $I(t_*)$ by its projection $I^{\diamond}(t_*)$ onto the φ axis and sets $I_\varphi(t_*)$ on the x, y plane

$$I(t_*) = \bigcup_{\varphi \in I^{\diamond}(t_*)} [I_\varphi(t_*) \times \{\varphi\}]. \qquad (2.2)$$

Similarly, we write the prediction set as

$$G(t^*) = \bigcup_{\bar\varphi \in G^{\diamond}(t^*)} [G_{\bar\varphi}(t^*) \times \{\bar\varphi\}]. \qquad (2.3)$$

Taking into account formulas (1.1) and (2.3), we can rewrite formula (2.1) as follows:

$$I(t^*) = \bigcup_{\bar\varphi \in G^{\diamond}(t^*)} [(G_{\bar\varphi}(t^*) \cap H^{\#}(t^*)) \times \{\bar\varphi\}]. \qquad (2.4)$$

Expression (2.4) is written in the same form as (2.2). This follows from the fact that

$$I_{\bar\varphi}(t^*) = G_{\bar\varphi}(t^*) \cap H^{\#}(t^*),$$

$$I^{\diamond}(t^*) = \{\bar\varphi \in G^{\diamond}(t^*): G_{\bar\varphi}(t^*) \cap H^{\#}(t^*) \neq \varnothing\}.$$

The representation of sets $I(t_*)$, $G(t^*)$, and $I(t^*)$ in form (2.2)–(2.4) allows us to pass below to a grid in the φ coordinate and consider only the sections corresponding to the nodes of the grid.

2.3. Consideration of Specific Features of Motion Dynamics

Let us show how sets $G^{\diamond}(t^*)$ and $G_{\bar\varphi}(t^*)$ can be determined on the basis of sets $I^{\diamond}(t_*)$ and $I_\varphi(t_*)$.

We fix control $u(\cdot)$ on the half-open interval $[t_*, t^*)$. Consider the prediction set $G(t^*, u(\cdot))$ for the initial set $I(t_*)$ and control $u(\cdot)$. We represent this set, as well as set $G(t^*)$, by projection $G^{\diamond}(t^*, u(\cdot))$ onto the φ axis and by sections $G_{\bar\varphi}(t^*, u(\cdot))$.

Phase coordinates x and y are absent in the right-hand side of system (0.1). Therefore, the third equation can be integrated independent of the first two equations:

$$\varphi(t) = \varphi(t_*) + \int_{t_*}^{t} ku(\tau)/V d\tau.$$

Hence,

$$G^{\diamond}(t^*, u(\cdot)) = \left\{\varphi + \int_{t_*}^{t^*} ku(t)/V dt : \varphi \in I^{\diamond}(t_*)\right\}.$$

We additionally fix a value $\varphi \in I^{\diamond}(t_*)$. The integration of the first two equations of system (0.1) for initial states from $I_\varphi(t_*)$ means that every point is shifted by the same vector. Setting

$$\bar\varphi = \varphi(t^*) = \varphi + \int_{t_*}^{t^*} ku(\tau)/V d\tau, \qquad (2.5)$$

we obtain

$$G_{\bar\varphi}(t^*, u(\cdot)) = I_\varphi(t_*) + \left(\int_{t_*}^{t^*} V\cos\varphi(t)dt, \int_{t_*}^{t^*} V\sin\varphi(t)dt\right)^{\mathrm{T}}. \qquad (2.6)$$

Here and below, superscript T denotes the transposition.

Taking over the values $\varphi \in I^{\diamond}(t_*)$, we obtain by formula (2.6) all sections $G_{\bar\varphi}(t^*, u(\cdot))$ of the prediction set $G(t^*, u(\cdot))$.

Assume that sets $G_{\bar\varphi}(t^*, u(\cdot))$ are given for any value of $\bar\varphi$. If $\bar\varphi \in G^{\diamond}(t^*, u(\cdot))$, i.e., by virtue of (2.5), $\bar\varphi$ corresponds to a certain $\varphi \in I^{\diamond}(t_*)$, then the section is defined by formula (2.6); otherwise, $G_{\bar\varphi}(t^*, u(\cdot)) = \varnothing$.

Now, we shall not fix a control $u(\cdot)$. The set $G^{\diamond}(t^*)$ is the attainability set by the third equation of system (0.1) for the initial set $I^{\diamond}(t_*)$; i.e.,

$$G^{\diamond}(t^*) = \bigcup_{u(\cdot)} G^{\diamond}(t^*, u(\cdot)).$$

The sets $G_{\bar\varphi}(t^*)$ are determined by the formula

$$G_{\bar\varphi}(t^*) = \bigcup_{u(\cdot)} G_{\bar\varphi}(t^*, u(\cdot)). \qquad (2.7)$$

2.4. Convexization of Sections of the Prediction Set

While determining sections $G_{\bar\varphi}(t^*)$, we deal with the union of sets (2.7) on the x, y plane. Then, sets

$G_{\bar{\varphi}}(t^*)$, $\bar{\varphi} \in G^{\diamond}(t^*)$ are intersected with set $H^{\#}(t^*)$. The difficulty is that sets $G_{\bar{\varphi}}(t^*)$ are nonconvex.

We make the convention that each set $G_{\bar{\varphi}}(t^*)$, $\bar{\varphi} \in G^{\diamond}(t^*)$, is replaced with its convex hull. Thus, simultaneously with operation (2.7) of taking the union over $u(\cdot)$ of sets $G_{\bar{\varphi}}(t^*, u(\cdot))$, we determine the convex hull. We obtain the set

$$\mathbf{G}_{\bar{\varphi}}(t^*) = \mathrm{conv}G_{\bar{\varphi}}(t^*).$$

Denoting

$$\mathbf{G}(t^*) = \bigcup_{\bar{\varphi} \in G^{\diamond}(t^*)} [\mathbf{G}_{\bar{\varphi}}(t^*) \times \{\bar{\varphi}\}],$$

$$\mathbf{I}(t^*) = \bigcup_{\bar{\varphi} \in G^{\diamond}(t^*)} [(\mathbf{G}_{\bar{\varphi}}(t^*) \cap H^{\#}(t^*)) \times \{\bar{\varphi}\}],$$

we have $\mathbf{G}(t^*) \supset G(t^*)$ and $\mathbf{I}(t^*) \supset I(t^*)$.

We make the convention to start the convexization from the initial instant t_0. Then, at instant t_*, we obtain a set $\mathbf{I}(t_*)$ with convex sections $\mathbf{I}_{\varphi}(t_*)$ Hence, determining set $\mathbf{G}_{\bar{\varphi}}(t^*)$, we construct the convex hull of a union of convex sets.

Replacing t^* in the above formulas for $\mathbf{G}(t^*)$ with an arbitrary $t \in [t_*, t^*]$, we obtain set $\mathbf{G}(t)$ at instant t. Calculating $\mathbf{G}(\tilde{t})$ for an instant $\tilde{t} \in (t_*, t^*)$, one can similarly find set $\mathbf{G}(\hat{t}; \tilde{t}, \mathbf{G}(\tilde{t}))$ constructed to instant $\hat{t} \in (\tilde{t}, t^*]$ from the initial set $\mathbf{G}(t)$. Obviously, $\mathbf{G}(\hat{t}; \tilde{t}, \mathbf{G}(\tilde{t})) \supset \mathbf{G}(\hat{t})$. However, the above-mentioned specific property of system (0.1) (the absence of phase coordinates x and y in the right-hand side) implies the stronger relation $\mathbf{G}(\hat{t}; \tilde{t}, \mathbf{G}(\tilde{t})) = \mathbf{G}(\hat{t})$. Thus, the mapping $t \longrightarrow \mathbf{G}(t)$ has the semigroup property.

Below, the construction of sets $\mathbf{G}(t)$ and $\mathbf{I}(t)$ is considered. Omitting the term *extended*, we still call them (as well as $G(t)$ and $I(t)$) the prediction set and the information set.

3. PRACTICAL CONSTRUCTION OF INFORMATION SETS

While numerically determining the information sets, we use a certain discretization of the constructions. We describe some elements of the discretization.

3.1. Discretization in t, u, and φ

Suppose that, at instant t_*, the information set $\mathbf{I}(t_*)$ is constructed and it is required to construct set $\mathbf{I}(t^*)$ at the instant $t^* > t_*$ of the next reading.

First, we find the prediction set $\mathbf{G}(t^*)$. For this purpose, we can partition the time interval $[t_*, t^*]$ with a step Δ and consider piecewise-constant controls $u(\cdot)$ that take either zero value or an extreme one: $[t^{(i)}, t^{(i+1)})$ on any half-open interval $\{-1, 0, 1\}$ of the partition obtained. The choice of these values is determined by the fact that the extremal motions that form the boundary of the attainability set satisfy [12] the Pontryagin maximum principle and, therefore, are generated for system (0.1) with the help of extreme control actions $u = \pm 1$. The value $u = 0$ is realized on the degenerate parts of the extremal motions. Taking over the piecewise-constant controls and executing the convexization operation at instant t^*, we approximately determine set $\mathbf{G}(t^*)$.

However, it is more convenient, taking into account the semigroup property of the mapping $t \longrightarrow \mathbf{G}(t)$, to recurrently recalculate set $\mathbf{G}(t)$ with step Δ up to instant t^*. In this case, at each Δ step, we use the three constant controls $u = -1, 0, 1$. Each step is completed with the convexization operation.

We construct the prediction set $\mathbf{G}(t^{(i+1)})$ at instant $t^{(i+1)}$ on the basis of set $\mathbf{G}(t^{(i)})$ constructed at instant $t^{(i)}$. We formally assume that $\mathbf{G}(t^{(1)}) = \mathbf{I}(t_*)$ at instant $t^{(1)} = t_*$.

We recalculate the set $\mathbf{G}(t^{(i)})$ into the set $\mathbf{G}(t^{(i+1)})$ in the following way. At instant $t^{(i)}$, we have a collection of nodes in φ. Each node is associated with a convex set $\mathbf{G}_{\varphi}(t^{(i)})$. Using controls $u = -1, 0, 1$, at instant $t^{(i+1)}$, we obtain three nodes of a new grid: $\bar{\varphi} = \varphi + \Delta ku/V$. The sections corresponding to these nodes have the form

$$\mathbf{G}_{\bar{\varphi}}(t^{(i+1)}, u) = \begin{cases} \mathbf{G}_{\varphi}(t^{(i)}) + \dfrac{V^2}{ku}(\sin\bar{\varphi} - \sin\varphi, \cos\varphi - \cos\bar{\varphi})^{\mathrm{T}}, \\ \text{if } u = \pm 1; \\ \mathbf{G}_{\varphi}(t^{(i)}) + \Delta V(\cos\varphi, \sin\varphi)^{\mathrm{T}}, \text{ if } u = 0. \end{cases}$$

Thus, the number of nodes of the new grid at step Δ can increase by three times. However, it occurs that some nodes are close. This allows us to paste together these nodes in order to diminish the total number of sections. Instead of any group of sections with close values of $\bar{\varphi}$, we introduce one section with the average value of the φ coordinate, which is the convex hull of the union of the sections of the group considered. The totality of the resultant sections composes the set $\mathbf{G}(t^{(i+1)})$.

Continuing in this way until instant t^* we obtain the set $\mathbf{G}(t^*)$.

When, at instant t^*, a reading occurs, we form the uncertainty set $H(t^*)$. This set is cylindrical in the φ coordinate and is completely determined by its projection $H^{\#}(t^*)$ onto the x, y plane.

Information set $\mathbf{I}(t^*)$ is obtained by intersecting each section $\mathbf{G}_{\varphi}(t^*)$ of the prediction set $\mathbf{G}(t^*)$ with the set $H^{\#}(t^*)$. The results of nonempty intersections compose the required set $\mathbf{I}(t^*)$.

3.2. Approximation by Polyhedrons

We deal with the convex sets $\mathbf{G}_\varphi(t)$, $\mathbf{I}_\varphi(t)$, and $H^{\#}(t)$ in the x, y plane. To represent these sets, we use convex polyhedrons, approximating them from above.

We fix a collection of m unit vectors $n_1, \ldots, n_m$ (a grid of normal vectors) located uniformly in angle in the x, y plane. We count these vectors counterclockwise from the x axis.

For any bounded closed set D in the plane, as its upper approximation, we take a polyhedron D_m determined by the set of values ρ_i of the support function of set D on vectors n_i:

$$D_m = \{(x, y)^{\mathrm{T}} : xn_{ix} + yn_{iy} \le \rho_i,\ i = 1, \ldots, m\}.$$

For $m = 4$, we have four vectors and the approximating polyhedron is a rectangle whose sides are parallel to the coordinate axes. The use of only logical operations for dealing with rectangles distinguishes this case from the standpoint of minimization of expenses of memory and computation time.

3.3. Construction of the Convex Hull of the Union

Let convex polyhedrons A and B be specified by vectors $\rho_1^A, \ldots, \rho_m^A$ and $\rho_1^B, \ldots, \rho_m^B$ of the support function values on a fixed grid of m vectors n_i. The convex hull $\mathrm{conv}(A \cup B)$ of the union of these polyhedrons is a polyhedron whose normal vectors do not necessarily belong to grid $n_1, \ldots, n_m$. We approximate the convex hull $\mathrm{conv}(A \cup B)$ from above by a convex polyhedron C determined by the set $\rho_1^C, \ldots, \rho_m^C$, where

$$\rho_i^C = \max(\rho_i^A, \rho_i^B), \quad i = 1, \ldots, m.$$

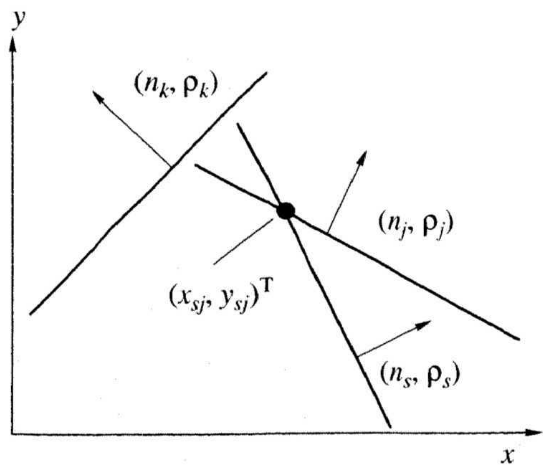

Fig. 2. Clarification of the intersection procedure.

Polyhedron C is minimal with respect to inclusion among the polyhedrons containing the union $A \cup B$ and specified by the values of the support function on the vectors $n_1, \ldots, n_m$.

3.4. Intersection Operation

The intersection of convex polyhedrons A and B specified by m-tuples $\rho_1^A, \ldots, \rho_m^A$ and $\rho_1^B, \ldots, \rho_m^B$ of values of the support functions is found by calculating m-tuple $\rho_1, \ldots, \rho_m$, where

$$\rho_i = \min(\rho_i^A, \rho_i^B), \quad i = 1, \ldots, m.$$

This m-tuple determines the polyhedron $C = A \cap B$ as the intersection of the half-planes $(xn_{ix} + yn_{iy}) \le \rho_i$, $i = 1, \ldots, m$.

The obtained m-tuple $\rho_1, \ldots, \rho_m$ does not necessarily coincide with m-tuple $\rho_1^C, \ldots, \rho_m^C$ of values of the support function of polyhedron C specified on the considered fixed grid of vectors n_i. Moreover, the intersection can be empty. To obtain the required m-tuple $\rho_1^C, \ldots, \rho_m^C$ of values of the support function of polyhedron C on the given fixed grid of vectors, we perform an additional processing of m-tuple $\rho_1, \ldots, \rho_m$. This processing consists of two steps.

Step 1. We consider m-tuple $\rho_1, \ldots, \rho_m$ as a cycled list in which elements ρ_1 and ρ_m are treated as neighbors. We delete from this list all elements ρ_i that do not coincide with the corresponding values ρ_i^C of the support function of set $C = A \cap B$.

Here, we use the following criterion. Let ρ_s, ρ_j, and ρ_k be three consecutive elements from the list considered. We find the point $(x_{sj}, y_{sj})^{\mathrm{T}}$ of the intersection of the boundaries of the half-planes specified by the pairs (n_s, ρ_s) and (n_j, ρ_j); i.e.,

$$(x_{sj}, y_{sj})^{\mathrm{T}} = \left(\frac{\rho_s n_{jy} - \rho_j n_{sy}}{n_{sx} n_{jy} - n_{jx} n_{sy}}, \frac{\rho_j n_{sx} - \rho_s n_{jx}}{n_{sx} n_{jy} - n_{jx} n_{sy}}\right)^{\mathrm{T}}.$$

If the point $(x_{sj}, y_{sj})^{\mathrm{T}}$ does not belong to the half-plane determined by the pair (n_k, ρ_k), i.e., if

$$x_{sj} n_{kx} + y_{sj} n_{ky} > \rho_k,$$

then we delete the middle element ρ_j. If the point $(x_{sj}, y_{sj})^{\mathrm{T}}$ belongs (Fig. 2) to this half-plane, i.e., if the relation

$$x_{sj} n_{kx} + y_{sj} n_{ky} \le \rho_k \tag{3.1}$$

holds, then we proceed to the consideration of the next triple.

We continue this process until we obtain an m-tuple such that condition (3.1) holds for any triple of neighboring vectors.

If, after deleting an element ρ_j, it occurs that the angle between neighboring vectors n_s and n_k is greater than or equal to π, i.e., if the condition

$$n_{sx}n_{ky} - n_{kx}n_{sy} \le 0 \qquad (3.2)$$

holds, then the intersection is an empty set and the execution of the intersection operation is terminated. In practical implementations of this algorithm, condition (3.2) should be tested with something to spare.

After completing the first step, we obtain a list $\{\rho_i^C\}$, every element of which is associated with a side of polyhedron C.

Step 2. We complete the list $\{\rho_i^C\}$ with the absent values of the support function on the fixed grid of vectors $n_1, \ldots, n_m$ and obtain the desired representation of the set $C = A \cap B$ in the form of the m-tuple $\rho_1^C, \ldots, \rho_m^C$. For completing the list, we use the following method.

Suppose that, between neighboring elements ρ_s and ρ_k, there are absent values of the support function.

We calculate the vertex $(x, y)^T$ of the polyhedron C corresponding to the sides with normal vectors n_s and n_k. We calculate the absent values ρ_j of the support function at the intermediate vectors n_j (between n_s and n_k) by the formula

$$\rho_j = xn_{jx} + yn_{jy}.$$

In the case of $m = 4$, the additional processing (Steps 1 and 2) of the 4-tuple ρ_1, ρ_2, ρ_3, ρ_4 is reduced only to checking the nonemptyness of the intersection of the sets A and B and the simultaneous validity of the two inequalities $\rho_1 + \rho_3 > 0$ and $\rho_2 + \rho_4 > 0$.

Algorithms similar to the described algorithm for constructing the support function of the intersection of polyhedrons in the plane are widely used in computational practice [15, 16].

4. COMPARISON WITH EXACT CONSTRUCTIONS

To make clear the character of errors introduced by the convexization operation applied while constructing prediction sets $\mathbf{G}(t)$, we compare the set $\mathbf{G}(t)$ with the exact prediction set (or, analogously, with the exact attainability set) $G(t)$. We assume that, at the instant $t_0 = 0$, the initial set is a point in the three-dimensional space $\{x, y, \varphi\}$; i.e., at the initial instant, the geometrical position and velocity direction are specified.

In paper [17], some formulas describing the boundary of the projection of the attainability set $G(t)$ of system (0.1) onto the x, y plane are presented. This projection will be denoted as $G^\#(t)$. The projection of the prediction set $\mathbf{G}(t)$ constructed by the algorithm proposed in this paper with the convexization at each Δ step onto the x, y plane will be denoted as $\mathbf{G}^\#(t)$. Compare the sets $G^\#(t)$ and $\mathbf{G}^\#(t)$. Without loss of generality, we assume that the initial geometrical position and the initial angle of the velocity vector are equal to zero.

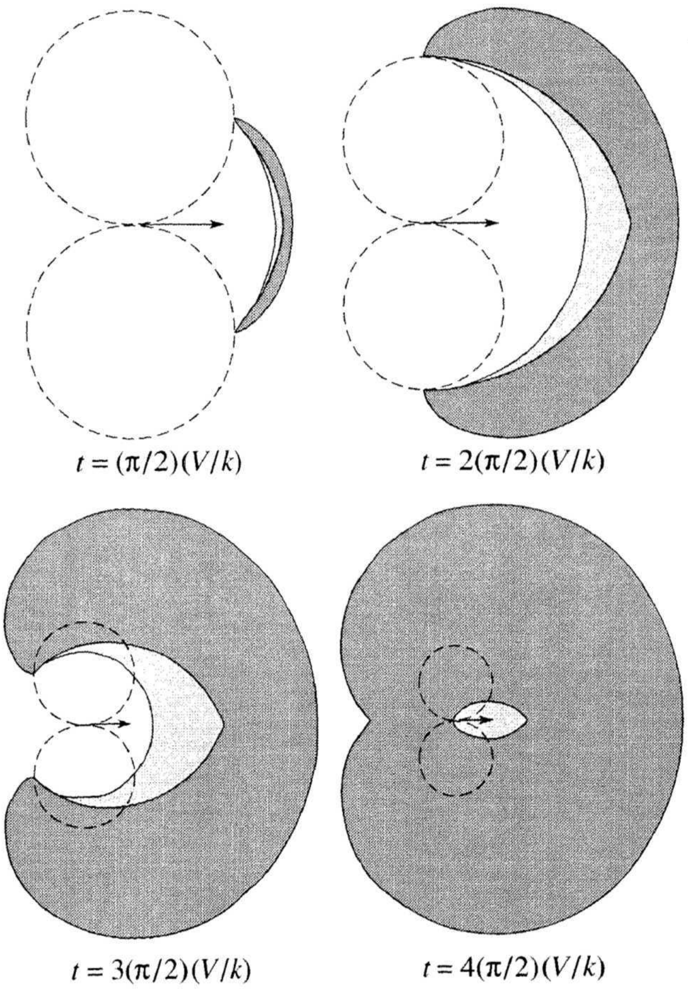

Fig. 3. Comparison with the exact attainability set in the projection onto the x, y plane.

In Fig. 3, the sets $G^\#(t)$ and $\mathbf{G}^\#(t)$ constructed for the instants $t = q(\pi/2)(V/k)$, $q = 1, 2, 3, 4$, are presented. These instants correspond to the rotation time of the velocity vector to the angle $q(\pi/2)$ for a motion with the maximal lateral acceleration. For each instant, we use a particular image scale. The motion trajectories with extreme controls $u = -1$ and 1 are circumferences with radii V^2/k. The initial velocity vector is shown as an arrow. The sets $G^\#(t)$ are shown by contours and dark filling, and the sets $\mathbf{G}^\#(t)$ have light filling. The sets $\mathbf{G}^\#(t)$ are constructed with a small time step Δ, and the number of normal vectors in the polyhedrons is set equal to $m = 64$. A rather dense φ grid is used in the interval $[-2\pi, 2\pi]$. We do not apply identification modulo 2π. One can see that the *external* parts of the boundaries of sets $G^\#(t)$ and $\mathbf{G}^\#(t)$ practically coincide and the *internal* parts are different. The inclusion $G^\#(t) \subset \mathbf{G}^\#(t)$ holds.

Figure 4 demonstrates the dependence of the constructed set $\mathbf{G}^\#(t)$ on the number of normal vectors. The

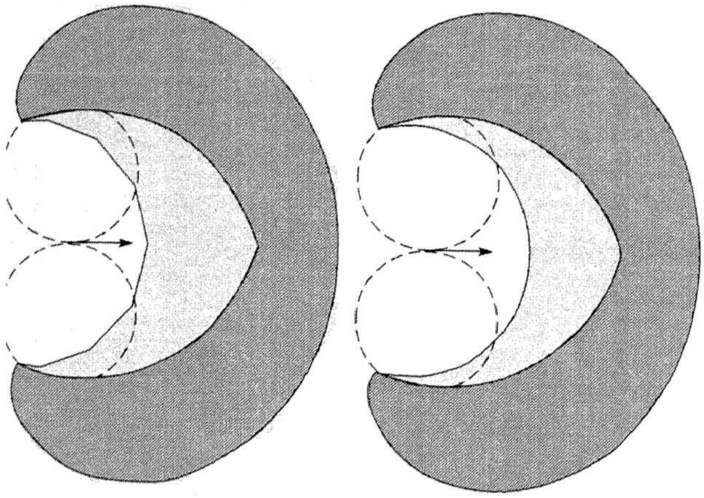

Fig. 4. The influence of the number of normal vectors on the construction of set $\mathbf{G}^{\#}(t)$.

nstructions are performed for the instant t = $\pi/4)(V/k)$ and for the number of normal vectors m = 2 and 24. The picture does not significantly change if e number of m normal vectors exceeds 24.

Let us proceed to comparing the three-dimensional ts $G(t)$ and $\mathbf{G}(t)$. The following theorem [18] holds.

Theorem. One can arrive at any point of the boundary of the attainability domain $G(t)$ of system (0.1) with the help of a piecewise control $u(\cdot)$ with at most two switchings. Moreover, in the case of two switchings, the following six versions of the control sequences are sufficient:

$$\begin{array}{lll} 1)\ 1, 0, 1; & 2)\ -1, 0, 1; & 3)\ 1, 0, -1; \\ 4)\ -1, 0, -1; & 5)\ 1, -1, 1; & 6)\ -1, 1, -1. \end{array} \quad (4.1)$$

This theorem allows us to construct the boundary of the attainability set $G(t)$ in the three-dimensional phase space with a good accuracy.

The attainability set $G(t)$ and the upper estimate $\mathbf{G}(t)$ for this set calculated by the algorithms for constructing the prediction set are shown in Figs. 5–7 for three instants $t = h\pi(V/k)$, $h = 2, 3, 4$ (for each instant, a particular scale is chosen). The thin lines show certain φ sections. While calculating the sets $G(t)$ and $\mathbf{G}(t)$, we do not identify the values of the φ coordinate modulo 2π.

We emphasize that, while constructing the set $\mathbf{G}(t)$, the exact attainability set $G(t)$ is supposed to be unknown. One can see that the errors caused by the convexization operation are noticeable only on one side of the surface bounding the exact set. In essence, any φ

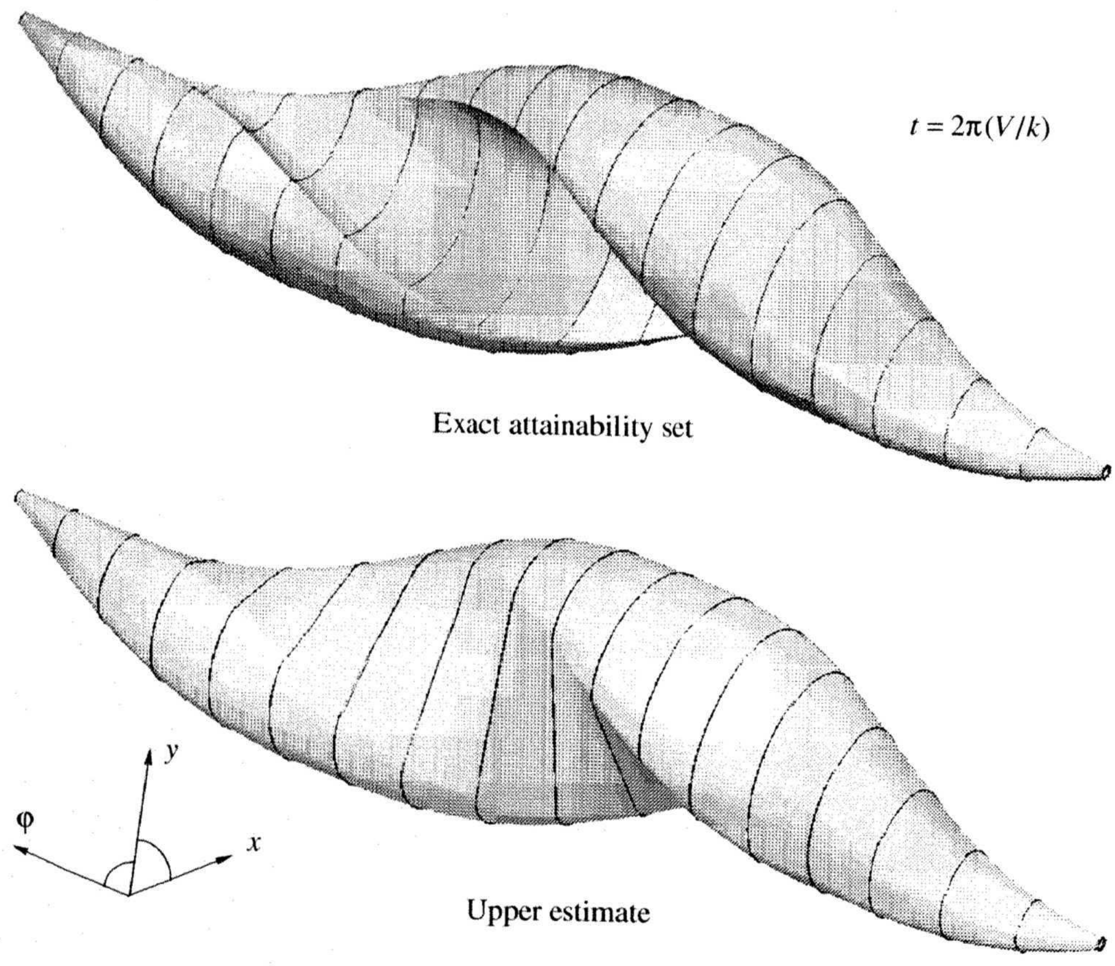

Fig. 5. Comparison with the exact attainability set, $t = 2\pi(V/k)$.

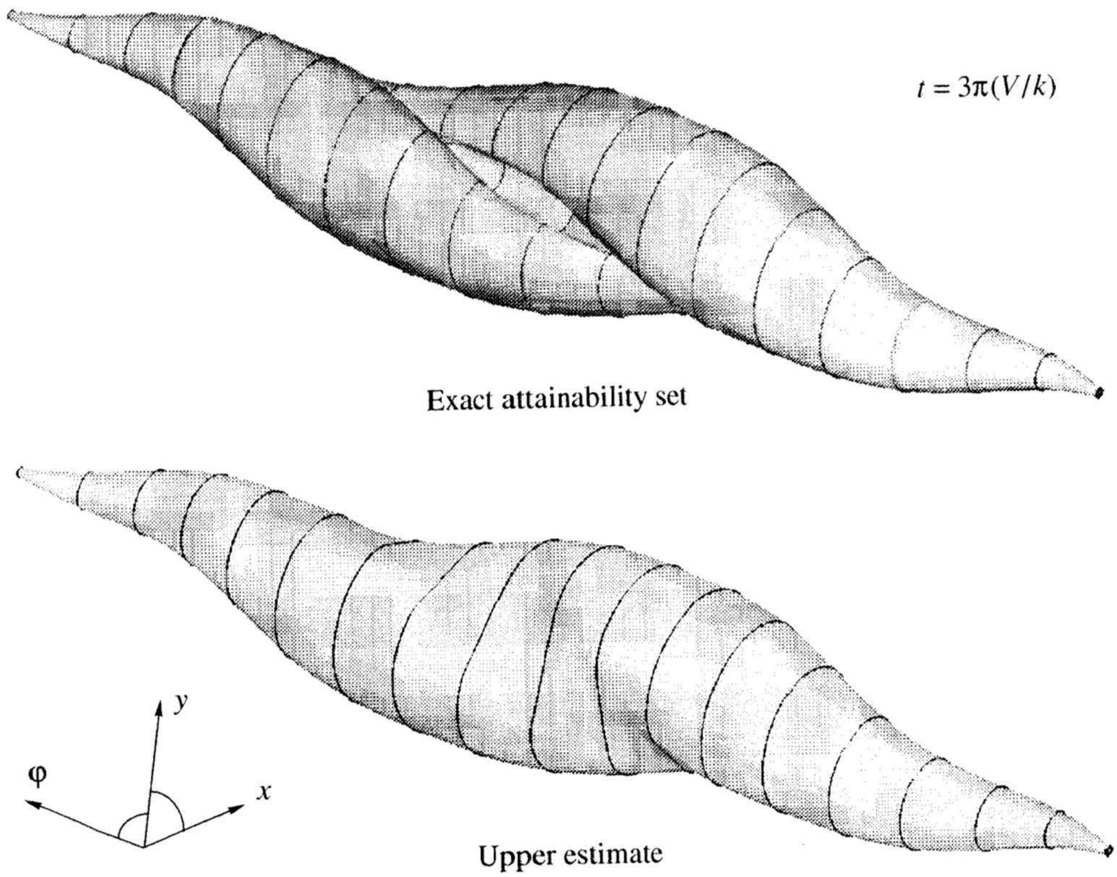

Fig. 6. Comparison with the exact attainability set, $t = 3\pi(V/k)$.

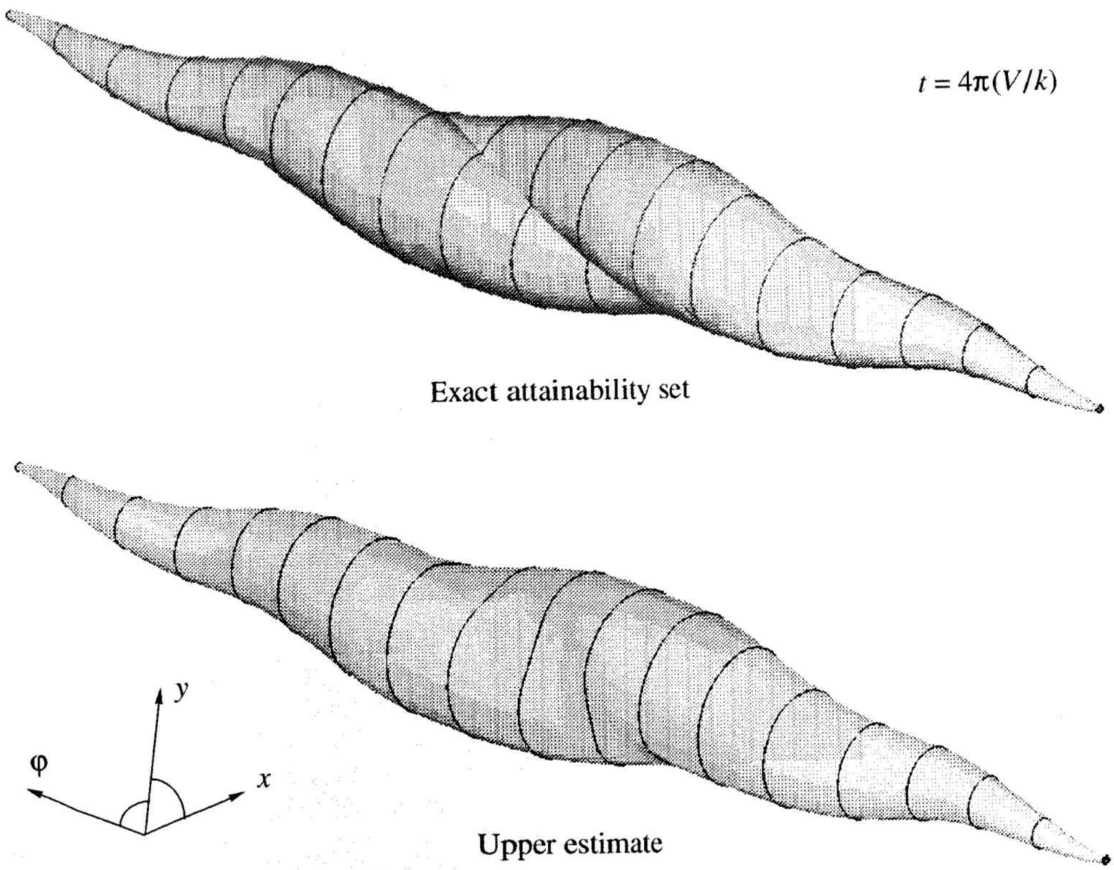

Fig. 7. Comparison with the exact attainability set, $t = 4\pi(V/k)$.

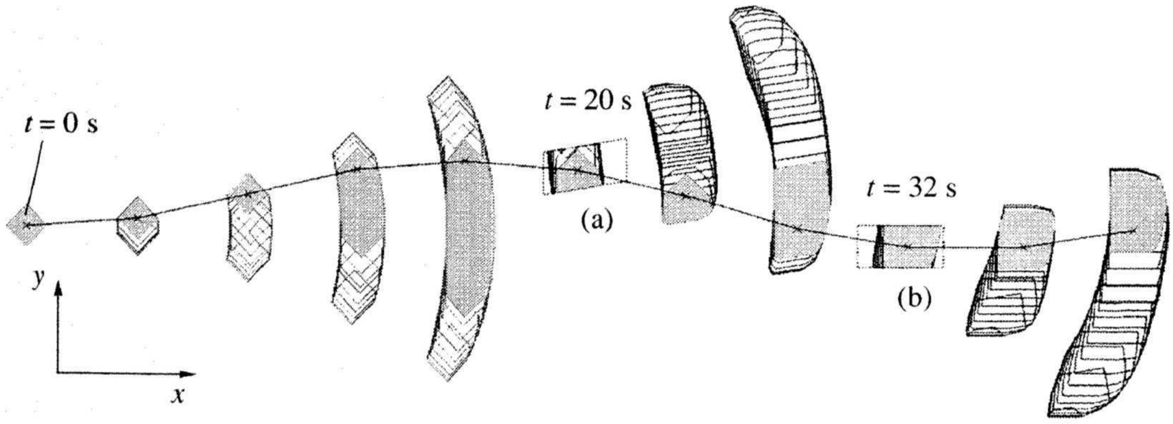

Fig. 8. The motion of the information set in the projection onto the x, y plane.

section of the set $\mathbf{G}(t)$ coincides with the convex hull of the corresponding φ section of the set $G(t)$.

In problems with incomplete information, one has to construct a three-dimensional prediction set for a quite arbitrary initial set. Although the algorithm described in this paper for constructing the prediction set does not produce the exact attainability set, it gives an upper estimate for this set and is rather simple to implement.

5. SIMULATION RESULTS FOR MOTION OF INFORMATION SETS

For the first example, we take the following parameters: $V = 400$ m/s, $k = 15$ m/s^2, $\Delta = 1$ s, $m = 60$. The initial information set $\mathbf{I}(0)$ has only one φ section in the form of a square. Thus, it is assumed that, at the initial instant $t_0 = 0$, the velocity direction is known. For $t \geq 0$, the proper motion of the phase point in the three-dimensional space is specified. The readings are formed with respect to this motion. The uncertainty sets are referred to these readings with a specified form of projections $H^{\#}(t)$ onto the x, y plane (no explicit constraints are imposed on the reading errors).

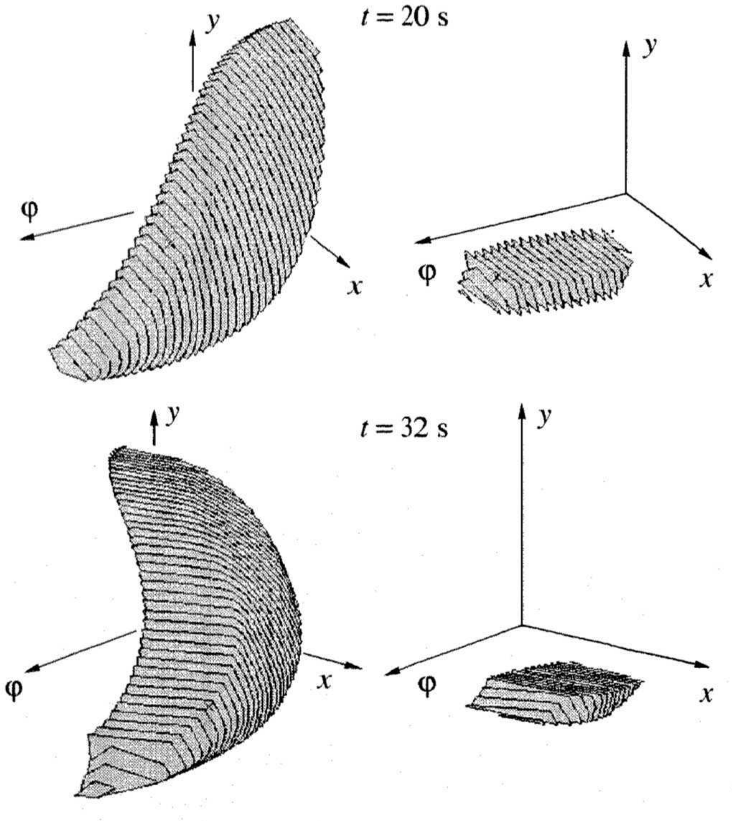

Fig. 9. The information set before and after the arrival of a reading.

In Fig. 8, we show the projection of the general picture of the information set variation on the time interval 0–40 s onto the x, y plane. The trajectory of the proper motion is shown by the solid line. Readings occur at instants 20 and 32 s. We assume that the projections $H^{\#}(20)$ and $H^{\#}(32)$ of the uncertainty sets are a parallelogram (a) and a rectangle (b). For the sake of clarity, we do not represent all sections in the information set, but represent only every second one. For the same reasons, we show the information sets only at each fourth time step: $\mathbf{I}(0)$, $\mathbf{I}(4)$, $\mathbf{I}(8)$, …, $\mathbf{I}(40)$. By crosses, we mark the actual point positions at the same instants. The sections of information sets that are closest in φ to the corresponding actual values are selected.

In Fig. 9, the information sets at instants 20 and 32 s are presented in more detail in the three-dimensional space. For each instant, two sets are shown: the set before the reading is taken into account (the prediction set) and the set with the reading taken into account.

In the example considered, the interval of φ values employed for the construction of the prediction set is less than 2π. Therefore, it is not necessary to identify the values of φ modulo 2π.

The simulation results for the case where the φ sections of information sets are specified in the form of rectangles whose sides are directed along the x and y axes are presented in Fig. 10. The information sets are constructed in the time interval of length 80 s. The following parameters are used: $V = 200$ m/s, $k = 5$ m/s^2, and $\Delta = 1$ s. The readings occur with time period 20 s; the corresponding uncertainty sets have the form of a square with side 400 m. The initial information set is

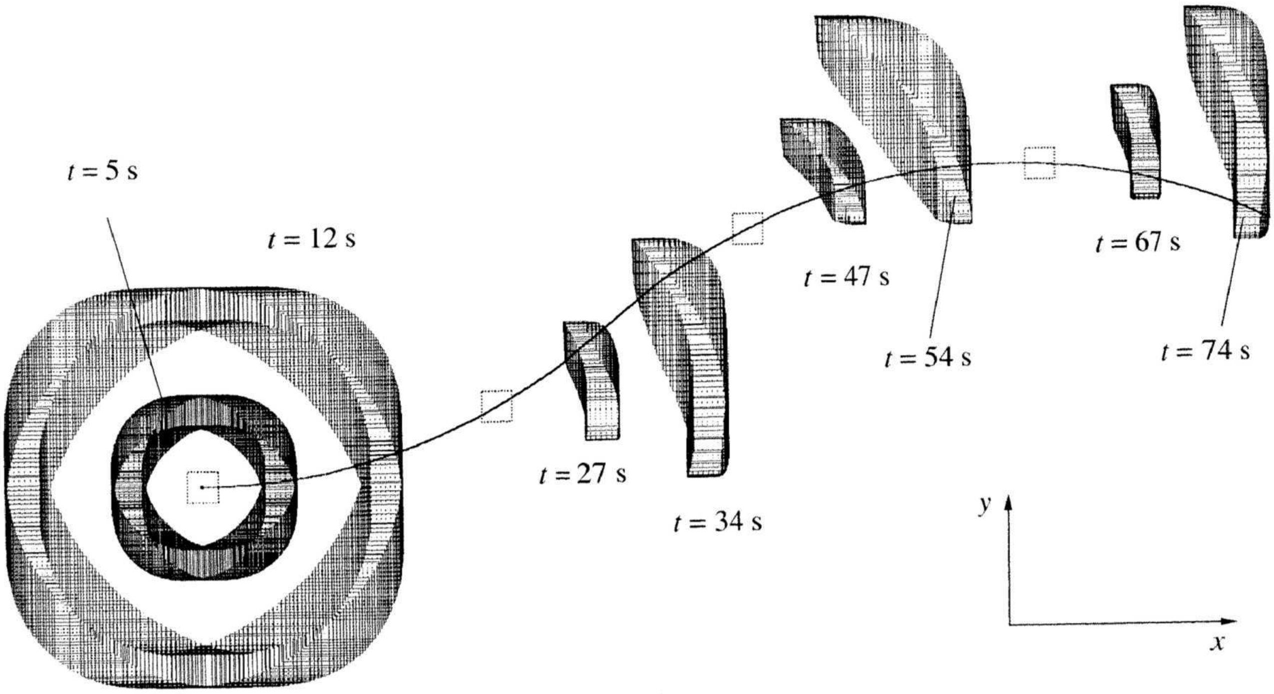

Fig. 10. The dynamics of the information set variation in the projection onto the x, y plane. The case of rectangular φ sections.

formed on the basis of the uncertainty set of the first reading and consists of 360 equal φ sections in the half-open interval $[0, 2\pi)$. Calculating the prediction sets, we identify the values of φ modulo 2π.

The trajectory of the proper motion has two turns in different directions. The uncertainty sets of the arriving readings are shown by dashed lines. The projections of the information sets onto the x, y plane are shown for instants 5, 12, 27, 34, 47, 54, 67, and 74 s. The projection of the initial information set coincides with the uncertainty set of the reading at the initial instant. Until the second reading occurs, the uncertainty in φ is retained in the half-open interval $[0, 2\pi)$. Therefore, the projections of the corresponding information sets onto the x, y plane have the form of a ring.

CONCLUSIONS

In the framework of the model problem of observing aircraft motion in a horizontal plane, a method for the upper estimation of the set of all possible phase states consistent with obtained readings and known constraints on the reading errors is proposed. Some results of numerical simulations characterizing the method errors are given.

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