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SWITCHING SURFACES IN LINEAR DIFFERENTIAL GAMES

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ABSTRACT. In this paper, we consider linear (in dynamics) conflict control problems (linear antagonistic differential games) with a fixed instant of termination and a continuous terminal cost function. We formulate and prove assertions on sufficient conditions under which such a method guarantees the obtaining of a result close to optimal by the minimizing player and has the stability property. In the concluding part of the paper, we give a brief description of publications devoted to computer modeling by using the proposed control method.

CONTENTS

Introduction

In engineering practice, control problems of motion in which components u_i, $i = \overline{1,k}$, of the vector control action are subjected to independent constraints $|u_i| \leq \mu_i$ are typical. For such problems, in finding a control optimizing a given performance index, methods based on the construction of switching surfaces in the state space are natural. Each such surface corresponds to its own component u_i of the control action, and at a current instant of time t, it divides the state space into two parts: to one side of the switching surface the component $u_i(t)$ assumes the value $-\mu_i$, and to the other side it assumes the value $+\mu_i$. In this case, problems of stability of the control method with respect to small errors in the construction of switching surfaces is important.

In this paper, we consider linear (in dynamics) conflict control problems (linear antagonistic differential games) with a fixed instant of termination and a continuous terminal cost function. The vector control action of the minimizing player is subjected to independent componentwise constraints $|u_i| \leq \mu_i$. We describe a method for constructing a feedback control using switching surfaces. We formulate and prove assertions on sufficient conditions under which such a method guarantees the obtaining of a result close to optimal by the minimizing player and has the stability property.

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In the concluding part of the paper, we give a brief description of publications devoted to computer modeling by using the proposed control method.

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Main Notation

(1)	superscript showing that the set considered (matrix, function, etc.) refers to the initial differential game;
(2)	superscript showing that the set considered (matrix, function, etc.) refers to the approximating differential game;
u	vector control action of the first player;
P	constraint on the control action of the first player;
k	number of scalar components of the control action of the first player;
μ_i	module of constraints on the scalar component with number i of the control of the first player;
μ	$= \sum\limits_{i=1}^{k} \mu_i$;
v'	vector control action of the second player;
Q	constraint on the control action of the second player;
ϑ	fixed instant of finishing the game;
γ	terminal cost function;
λ	Lipschitz constant of the cost function in the approximating game;
T	interval of the game;
Z	$= T \times \mathbb{R}^n$, game space;
U	strategy of the first player;
Δ	step of the discrete control scheme of the first player;
K	set of admissible program controls of the second player;
Γ	guarantee of the first player in the differential game;
$\boldsymbol{\Gamma}$	value function of the differential game;
$V^{(2)}$	u-stable function in the approximating game;
$W_c^{(2)}$	level set of the function $V^{(2)}$;
$\mathrm{Var}(V^{(2)}, [t_*, t^*])$	increment of the function $V^{(2)}$ on the closed interval $[t_*, t^*]$;
$B^{(3)}$	auxiliary matrix function defined on the interval T;
$B_i^{(3)}(t)$	column with number i of the matrix $B^{(3)}(t)$;
β_i	Lipschitz constant of the function $t \mapsto B_i^{(3)}(t)$;
β	maximum among the numbers β_i, $i = \overline{1, k}$;
σ_i	maximum of the module $\lvert B_i^{(3)}(t) \rvert$ on the interval T;
σ	maximum among the numbers σ_i, $i = \overline{1, k}$;
$\chi(t_*, t^*)$	integral characteristic of the difference between the dynamics of the initial and approximating games;
$\Pi(i, t)$	switching "surface" corresponding to instant t of the ith component of the control action;
$\Pi_-(i, t)$	the part of the space $\mathbb{R}^n$ lying to the negative side with respect to the "surface" $\Pi(i, t)$;
$\Pi_+(i, t)$	the part of the space $\mathbb{R}^n$ lying to the positive side with respect to the "surface" $\Pi(i, t)$;
$\mathbf{U}$	multivalued strategy of the first player defined on the basis of the sets $\Pi(i, t)$;
$\Pi^r(i, t)$	geometric r-neighborhood of the "surface" $\Pi(i, t)$;
$\mathbf{U}^r$	multivalued strategy of the first player defined on the basis of the sets $\Pi^r(i, t)$;
$\Pi^c(i, t)$	c-neighborhood of the "surface" $\Pi(i, t)$;

G	reachable set of the control system;
int	interior of a set;
C_k^h	binomial coefficient with k and h;
$\mathcal{G}(\mathcal{B})$	subspaces spanned by the set $\mathcal{B}$ of finitely many vectors from $\mathbb{R}^n$;
$q(F)$	the number of elements of a finite set F.

1. Statement of the Problem and Formulation of the Main Results

1.1. Preparatory description of the problem. Let the linear differential game with a fixed termination time ϑ be described by the relations

$$\begin{gathered} \dot{y}(t) = B^{(1)}(t)u(t) + C^{(1)}(t)v(t), \\ y(t) \in \mathbb{R}^n, \quad u(t) \in P^{(1)}, \quad v(t) \in Q^{(1)}; \quad \gamma^{(1)}(y(\vartheta)). \end{gathered} \tag{1.1}$$

Here, $y(t)$ is the state vector, $u(t)$ is the control action of the first player, $v(t)$ is that of the second, and the matrix functions $B^{(1)}$ and $C^{(1)}$ are piecewise continuous. It is assumed that the set $P^{(1)}$ bounding the control action of the first player is a "rectangular parallelepiped" in the space $\mathbb{R}^k$, i.e.,

$$P^{(1)} := \big\{u \in \mathbb{R}^k : |u_i| \le \mu_i,\ i = \overline{1,k}\big\}.$$

In addition,

$$\mu := \sum_{i=1}^{k} \mu_i > 0.$$

The set $Q^{(1)}$ bounding the control action of the second player is assumed to be a convex compact set in a finite-dimensional space. Let $\gamma^{(1)} : \mathbb{R}^n \mapsto \mathbb{R}$ be a continuous cost function. The first player minimizes the values $\gamma^{(1)}(y(\vartheta))$ of the cost function, whereas the interests of the second player are the opposite.

Game (1.1) is said to be initial. The notation referring to it is equipped with the superscript (1).

Let us agree that initial instants t_0 for game (1.1) belong to a closed interval $T = [\vartheta_1, \vartheta]$, where $\vartheta_1 < \vartheta$. Let

$$Z := T \times \mathbb{R}^n$$

be the game space.

An admissible open-loop control $u(\cdot)$ ($v(\cdot)$) of the first (second) player is a measurable function of time, $t \mapsto u(t)$ ($t \mapsto v(t)$), such that for any t, it satisfies the constraint $u(t) \in P^{(1)}$ ($v(t) \in Q^{(1)}$). Let $K^{(1)}$ be the set of all admissible open-loop controls $v(\cdot)$ of the second player.

Following [29], as admissible strategies of the first player, we consider arbitrary functions $U : (t,x) \mapsto U(t,x)$ defined on the set Z with values in $P^{(1)}$. We denote by $y^{(1)}(\cdot; t_0, x_0, U, \Delta, v(\cdot))$ the step-by-step motion of system (1.1) from the position (t_0, x_0) when the first player applies a strategy U in the discrete control scheme [29] with step $\Delta > 0$ and a control $v(\cdot)$ is realized for the second player. We set

$$\Gamma^{(1)}(t_0, x_0, U, \Delta) := \sup_{v(\cdot)\in K^{(1)}} \gamma^{(1)}(y^{(1)}(\vartheta; t_0, x_0, U, \Delta, v(\cdot))). \tag{1.2}$$

The quantity $\Gamma^{(1)}(t_0, x_0, U, \Delta)$ is a guarantee that the strategy U ensures to the first player for the initial position (t_0, x_0) in the discrete control scheme with step Δ. The best guarantee of the first player for the initial position (t_0, x_0) is defined by the formula

$$\boldsymbol{\Gamma}^{(1)}(t_0, x_0) := \min_U \varlimsup_{\Delta\to 0} \Gamma^{(1)}(t_0, x_0, U, \Delta), \tag{1.3}$$

where $\varlimsup$ means the upper limit. In [29], it is shown that minimum in U is attained. Note that, according to formulas (1.2) and (1.3), the dependence of the optimal strategy of the first player on the initial position (t_0, x_0) is not excluded.

It is known [29, 30] that the best guaranteed result $\mathbf{\Gamma}^{(1)}(t_0, x_0)$ coincides with the symmetrically defined best guaranteed result of the second player. Therefore, $\mathbf{\Gamma}^{(1)}(t_0, x_0)$ is also called the value of the value function at the point (t_0, x_0).

In this paper, we will show that, under certain additional assumptions in game (1.1), there exists a universal optimal strategy U^* of the first player stable with respect to the error of its numerical assignment.

The universality means that the strategy U^* is optimal for all initial positions $(t_0, x_0) \in Z$. It should be stressed that we are speaking about the universality in the "rigid" sense: the strategies considered are functions of only t, x. In the class of strategies additionally depending on a certain "accuracy parameter," the existence of optimal universal strategies was proved in [28] for a wide class of problems.

In this paper, we define the universal optimal strategy $(t, x) \mapsto U^*(t, x)$ using "switching surfaces." At each instant of time t, with each component u_i, $i = \overline{1, k}$, of the control action u, we associate its own switching surface. To one side of the switching surface the control u_i assumes the value $-\mu_i$, and to the other side it assumes the value $+\mu_i$. On the switching surface itself, the value of the control u_i can be arbitrarily chosen from the closed interval $[-\mu_i, \mu_i]$.

The problem on the existence of optimal universal strategies in differential games was briefly discussed in [29, p. 48], and it became topical after [42], in which the example of a game problem in which there is no universal strategy was presented. In [9, 10], it is shown that for linear differential games of the form (1.1), but in the case where the set $P^{(1)}$ is a segment (i.e., the control action u is scalar), there exists a stable universal strategy of the first (minimizing) player, and it can be given by using switching surfaces varying in time. In [46, 47, 49, 50], it was proved that if the set $Q^{(1)}$ is a segment (i.e., the control action v is, in fact, scalar), then there exists a universal optimal strategy of the second (maximizing) player, and it also can be given by the switching surfaces. However, such a strategy has no stability property. The influence of the loss of stability was demonstrated by using computer modeling in [49].

The constructions proposed in the paper generalize those described in [9, 10]. Another approach for proving the existence of the universal strategy in the case of a convex cost function was sketched in [2].

The use of switching surfaces for constructing a feedback control is very natural from the engineering point of view (for applications to game problems, see, e.g., [24]). The goal of the paper is to reveal the conditions under which, in the class of differential games considered, we obtain an optimal and stable control method.

As in [9, 10], we accept the following scheme of arguments. Orienting ourselves toward computer constructions, we replace the initial differential game by a convenient approximating game for which it is possible to construct a certain u-stable [29, 30] function or even the value function of the game. Processing such a function, we obtain switching surfaces. We apply the found switching surfaces in the initial differential game for defining the universal strategy of the first player. We estimate the guarantee of the first player that he ensures by applying the constructed universal strategy. As a consequence of such an estimate, we obtain the result concerning the existence of the universal optimal stable strategy in game (1.1).

This paper was published earlier as a preprint [37]. The result concerning the case of a scalar control of the first player was described in [36].

We make a comment on the writing of the dynamics of the linear differential game in the form (1.1). The specific character of this notation is that the state variables do not enter the right-hand side. Let the linear differential game with a fixed time of termination ϑ have the form

$$\dot{\mathbf{y}}(t) = \mathbf{A}(t)\mathbf{y}(t) + \mathbf{B}(t)u(t) + \mathbf{C}(t)v(t),$$
$$\mathbf{y}(t) \in \mathbb{R}^m, \quad u(t) \in P^{(1)}, \quad v(t) \in Q^{(1)}; \quad \gamma(\mathbf{y}(\vartheta)).$$

Assume that the cost function γ is determined only by values of certain n coordinates, $n \leq m$, of the state vector at the termination instant. Then the passage to (1.1) is performed [29, p. 160], [27, p. 354] by using the standard transformation

$$y(t) = X_{n,m}(\vartheta, t)\mathbf{y}(t),$$

where $X_{n,m}(\vartheta, t)$ is the $(n \times m)$-matrix composed of the corresponding n rows of the fundamental Cauchy matrix for the system $\dot{\mathbf{y}}(t) = \mathbf{A}(t)\mathbf{y}(t)$. In this case,

$$B^{(1)}(t) = X_{n,m}(\vartheta, t)\mathbf{B}(t), \quad C^{(1)}(t) = X_{n,m}(\vartheta, t)\mathbf{C}(t), \quad \gamma^{(1)}(y(\vartheta)) = \gamma(\mathbf{y}(\vartheta)).$$

1.2. Approximating game. Along with game (1.1), we consider one more game

$$\begin{gathered} \dot{y}(t) = B^{(2)}(t)u(t) + C^{(2)}(t)v(t), \\ y(t) \in \mathbb{R}^n, \quad u(t) \in P^{(2)} = P^{(1)}, \quad v(t) \in Q^{(2)}; \quad \gamma^{(2)}(y(\vartheta)) \end{gathered} \tag{1.4}$$

with a fixed time of termination ϑ. We will interpret game (1.4) as an approximation of game (1.1) that is convenient for calculations. Here, $y(t)$ is the state vector and the functions $B^{(2)}$ and $C^{(2)}$ are piecewise-continuous. The set $P^{(2)} = P^{(1)}$ bounding the control action of the first player is the same as in game (1.1), and the set $Q^{(2)}$ is a compact set in a finite-dimensional space. The cost function $\gamma^{(2)} : \mathbb{R}^n \mapsto \mathbb{R}$ is assumed to be Lipschitz with constant λ and satisfies the condition $\gamma^{(2)}(x) \to \infty$ as $|x| \to \infty$. The first player minimizes the values of $\gamma^{(2)}(y(\vartheta))$, and the second player maximizes them.

The belonging of one quantity or another to the approximating game is indicated by the superscript (2). Admissible controls $u(\cdot)$ and $v(\cdot)$ of the first and second players are defined similarly as for those for game (1.1). Denote by $K^{(2)}$ the set of all open-loop controls $v(\cdot)$ of the second player.

We assume that in the framework of the approximating game (1.4), a certain continuous u-stable function $V^{(2)} : Z \mapsto \mathbb{R}$ satisfying the boundary condition

$$V^{(2)}(\vartheta, x) = \gamma^{(2)}(x), \quad x \in \mathbb{R}^n,$$

is constructed. According to [29, 30], a function $V^{(2)}$ is said to be u-stable if for any position $(t_*, x_*) \in Z$, for any $t^* \in (t_*, \vartheta]$, and for any $v(\cdot) \in K^{(2)}$, there exists an admissible open-loop control $u(\cdot)$ of the first player such that for the motion $y^{(2)}(t) = y^{(2)}(t; t_*, x_*, u(\cdot), v(\cdot))$, the inequality $V^{(2)}(t^*, y^{(2)}(t^*)) \le V^{(2)}(t_*, x_*)$ holds.

Assume that the function $V^{(2)}$ is Lipschitz-continuous with constant λ. If $V^{(2)}$ is the value function of game (1.4), then the Lipschitz property follows [41, pp. 110–111] from the condition imposed on the function $\gamma^{(2)}$.

Let $B^{(3)}$ be a matrix function on T each of whose columns $B_i^{(3)}$, $i = \overline{1, k}$, satisfies the Lipshitz condition with constant β_i. Substantively, the function $B^{(3)}$ can be treated as a Lipschitz approximation of the functions $B^{(1)}$ and $B^{(2)}$. Denote

$$\beta := \max_{i=\overline{1,k}} \beta_i; \quad \sigma_i := \max_{t \in T} |B_i^{(3)}(t)|, \quad i = \overline{1, k}; \quad \sigma := \max_{i=\overline{1,k}} \sigma_i.$$

Assume that $\beta \ge 0$ and $\sigma > 0$.

1.3. Condition 1. Let us formulate the requirement on the function $V^{(2)}$, which then allows us to introduce switching surfaces.

Condition 1. For any $i = \overline{1, k}$ and any $t \in T$ such that $B_i^{(3)}(t) \neq 0$, the restriction of the function $V^{(2)}(t, \cdot)$ to any line in $\mathbb{R}^n$ parallel to the vector $B_i^{(3)}(t)$ is a function whose set of minimum points is a segment (possibly consisting of a single point) and which is strictly monotone to both sides of this segment.

In particular, Condition 1 holds if for any $t \in T$, the function $V^{(2)}(t, \cdot)$ is convex. In the case where $V^{(2)}$ is the value function of the approximating game (1.4), to ensure the convexity of the function $V^{(2)}(t, \cdot)$, $t \in T$, it suffices to require the convexity of the cost function $\gamma^{(2)}$.

1.4. Switching surfaces. Multivalued function $\mathbf{U}^0$. Introduce the following notation. For any $i = \overline{1,k}$ and $(t,x) \in Z$, we set

$$\mathcal{A}(i,t,x) := \{z \in \mathbb{R}^n : z = x + \alpha B_i^{(3)}(t),\ \alpha \in \mathbb{R}\}, \tag{1.5}$$

$$\mathcal{V}(i,t,x) := \min_{z \in \mathcal{A}(i,t,x)} V^{(2)}(t,z). \tag{1.6}$$

If $B_i^{(3)}(t) \neq 0$, then the set $\mathcal{A}(i,t,x)$ is the line in the space $\mathbb{R}^n$ passing through the point x parallel to the vector $B_i^{(3)}(t)$. In this case, $\mathcal{V}(i,t,x)$ is the minimum value of the function $V^{(2)}$ on the line $\mathcal{A}(i,t,x)$. The minimum is attained because of the continuity of the function $V^{(2)}(t,\cdot)$ and the fact that it tends to ∞ as $|x| \to \infty$. By Condition 1, the set of minimum points is a segment, which can consist of a single point. If $B_i^{(3)}(t) = 0$, then the set $\mathcal{A}(i,t,x)$ is degenerate and coincides with the point x. In this case, the value of $\mathcal{V}(i,t,x)$ coincides with $V^{(2)}(t,x)$.

Further, for all $i = \overline{1,k}$ and $t \in T$, let

$$\begin{aligned} \Pi(i,t) &:= \{x \in \mathbb{R}^n : V^{(2)}(t,x) = \mathcal{V}(i,t,x)\}, \\ \Pi_-(i,t) &:= \{x \in \mathbb{R}^n : x + \alpha B_i^{(3)}(t) \notin \Pi(i,t),\ \forall \alpha \geq 0\}, \\ \Pi_+(i,t) &:= \{x \in \mathbb{R}^n : x + \alpha B_i^{(3)}(t) \notin \Pi(i,t),\ \forall \alpha \leq 0\}. \end{aligned} \tag{1.7}$$

Therefore, the sets $\Pi_-(i,t)$, $\Pi(i,t)$, and $\Pi_+(i,t)$ are defined on the basis of the function $V^{(2)}(t,\cdot)$ and the vector $B_i^{(3)}(t)$. The sets $\Pi_-(i,t)$ and $\Pi_+(i,t)$ in the space $\mathbb{R}^n$ lie on different sides of the set $\Pi(i,t)$. It follows from Condition 1 that for any $(t,x) \in Z$, the function $V^{(2)}(t,\cdot)$ monotonically increases (monotonically decreases) in the direction of the vector $B_i^{(3)}(t)$ on the intersection of the line $\mathcal{A}(i,t,x)$ with the set $\Pi_-(i,t)$ $(\Pi_+(i,t))$.

For each $i = \overline{1,k}$, on Z, we define the scalar multivalued function

$$\mathbf{U}_i^0(t,x) := \begin{cases} \{-\mu_i\}, & x \in \Pi_-(i,t), \\ \{\mu_i\}, & x \in \Pi_+(i,t), \\ [-\mu_i, \mu_i], & x \in \Pi(i,t). \end{cases}$$

The function $\mathbf{U}_i^0(t,\cdot)$ assumes the extreme values from the segment $[-\mu_i, \mu_i]$ in the sets $\Pi_-(i,t)$ and $\Pi_+(i,t)$ and "switches" from one extreme value to another in the set $\Pi(i,t)$. Although the set $\Pi(i,t)$ is not a surface in the space $\mathbb{R}^n$ in the conventional sense, for clarity it will be called the switching surface for the ith component at the instant of time t.

On Z, we introduce the multivalued function

$$\mathbf{U}^0(t,x) := \begin{pmatrix} \mathbf{U}_1^0(t,x) \\ \mathbf{U}_2^0(t,x) \\ \vdots \\ \mathbf{U}_k^0(t,x) \end{pmatrix}.$$

1.5. Sets $\Pi^r(i,t)$. Multivalued function $\mathbf{U}^r$. Let us continue the introduction of notation for the formulation of the main result of the paper.

Let $r \geq 0$. In the case $B_i^{(3)}(t) \neq 0$, we set

$$\Pi^r(i,t) := \left\{ x \in \mathbb{R}^n : x = z + \alpha \frac{B_i^{(3)}(t)}{|B_i^{(3)}(t)|},\ z \in \Pi(i,t),\ |\alpha| \leq r \right\}.$$

The set $\Pi^r(i,t)$ is a geometric r-expansion of the set $\Pi(i,t)$. The expansion is performed by using the vector $B_i^{(3)}(t)$. The set $\Pi^r(i,t)$ will also be called the r-neighborhood of the surface $\Pi(i,t)$. If $B_i^{(3)}(t) = 0$, we take $\Pi^r(i,t) = \Pi(i,t) = \mathbb{R}^n$.

Introduce the sets

$$\Pi^r_-(i,t) := \{x \in \mathbb{R}^n : x + \alpha B_i^{(3)}(t) \notin \Pi^r(i,t),\ \forall \alpha \geq 0\},$$
$$\Pi^r_+(i,t) := \{x \in \mathbb{R}^n : x + \alpha B_i^{(3)}(t) \notin \Pi^r(i,t),\ \forall \alpha \leq 0\}.$$

The set $\Pi^r_-(i,t)$ ($\Pi^r_+(i,t)$) is the part of the space $\mathbb{R}^n$ located (with respect to $\Pi^r(i,t)$) to the (opposite) direction of the vector $B_i^{(3)}(t)$. Obviously,

$$\Pi^r_-(i,t) \subset \Pi_-(i,t), \quad \Pi^r_+(i,t) \subset \Pi_+(i,t).$$

For $r = 0$, we have $\Pi^r(i,t) = \Pi(i,t)$, $\Pi^r_-(i,t) = \Pi_-(i,t)$, and $\Pi^r_+(i,t) = \Pi_+(i,t)$.

For each $i = \overline{1,k}$, on Z, we introduce the scalar multivalued function

$$\mathbf{U}_i^r(t,x) := \begin{cases} \{-\mu_i\}, & x \in \Pi^r_-(i,t), \\ \{\mu_i\}, & x \in \Pi^r_+(i,t), \\ [-\mu_i, \mu_i], & x \in \Pi^r(i,t). \end{cases}$$

Further, on Z we define the vector multivalued function

$$\mathbf{U}^r(t,x) := \begin{pmatrix} \mathbf{U}_1^r(t,x) \\ \mathbf{U}_2^r(t,x) \\ \vdots \\ \mathbf{U}_k^r(t,x) \end{pmatrix}.$$

1.6. Condition 2. Let us formulate one more additional condition.

Let I be the set of subscripts $\overline{1,k}$, and let F be an arbitrary subset of I. For any $(t,x) \in Z$, we set

$$\begin{gathered} \mathcal{A}(F,t,x) := \left\{ z \in \mathbb{R}^n : z = x + \sum_{i \in F} \alpha_i B_i^{(3)}(t),\ \alpha_i \in \mathbb{R} \right\}, \\ \mathcal{V}(F,t,x) := \min_{z \in \mathcal{A}(F,t,x)} V^{(2)}(t,z). \end{gathered} \tag{1.8}$$

The set $\mathcal{A}(F,t,x)$ is a plane in the space $\mathbb{R}^n$ passing through the point x. The plane is generated by the set of vectors $B_i^{(3)}(t)$, $i \in F$, and its dimension is equal to the number of linearly independent vectors of this set. In formula (1.8), the minimum is attained by the properties of the function $V^{(2)}(t,\cdot)$. Moreover, the set of minimum points is a compact set.

For all $F \subset I$ and $t \in T$, let

$$\Pi(F,t) := \{x \in \mathbb{R}^n : V^{(2)}(t,x) = \mathcal{V}(F,t,x)\}.$$

It is easy to see that if $F_1 \subset F_2$, then $\Pi(F_2,t) \subset \Pi(F_1,t)$. In particular, for any $i \in F$, we have

$$\Pi(F,t) \subset \Pi(i,t).$$

Hence

$$\Pi(F,t) \subset \bigcap_{i \in F} \Pi(i,t).$$

The additional assumption on the problem consists of the requirement that the inverse inclusion hold.

Condition 2. For any $t \in T$ and any subset $F \subset I$, the following inclusion holds:

$$\bigcap_{i \in F} \Pi(i,t) \subset \Pi(F,t).$$

Condition 2 holds in the case where the set $P^{(1)}$ is a segment, i.e., in the case of a scalar control of the first player. In the non-scalar case, Condition 2 means that any point common for "individual" surfaces $\Pi(i,t)$, $i \in F$, lies in the set $\Pi(F,t)$.

We will also use the notation $\mathcal{A}(F,t,x)$, $\mathcal{V}(F,t,x)$, and $\Pi(F,t)$ for the case $F = \varnothing$. Let us agree that

$$\mathcal{A}(\varnothing,t,x) := x, \quad \mathcal{V}(\varnothing,t,x) := V^{(2)}(t,x), \quad \Pi(\varnothing,t) := \mathbb{R}^n.$$

A substantive explanation of this can be the following. Let us add the component u_{k_1} to the components u_i, $i = \overline{1,k}$, of the control action, but in this case we set

$$B^{(1)}_{k+1}(t) = B^{(2)}_{k+1}(t) = B^{(3)}_{k+1}(t) \equiv 0.$$

Then for F consisting of one subscript $k+1$, we have

$$\mathcal{A}(F,t,x) = x, \quad \mathcal{V}(F,t,x) = V^{(2)}(t,x), \quad \Pi(F,t) = \mathbb{R}^n.$$

Such F in the imaginary expanded set of components of the control action is equivalent to the empty set in dealing with the initial set of subscripts of components of the control action.

1.7. Formulation of the main results. For any instants t_* and t^* from the interval T, we set

$$\begin{aligned}\chi(t_*,t^*) := \sum_{i=1}^{k} \mu_i \int\limits_{t_*}^{t^*} \Big(\big|B^{(1)}_i(t) - B^{(3)}_i(t)\big| + \big|B^{(2)}_i(t) - B^{(3)}_i(t)\big| \Big)\,dt \\ + \int\limits_{t_*}^{t^*} \max_{\substack{\ell\in\mathbb{R}^n \\ |\ell|\le 1}} \left[\max_{q\in Q^{(1)}} \ell' C^{(1)}(t) q - \max_{q\in Q^{(2)}} \ell' C^{(2)}(t) q \right] dt.\end{aligned} \tag{1.9}$$

The quantity $\chi(t_*,t^*)$ characterizes (in the integral sense) the difference between the functions $B^{(1)}_i$, $B^{(2)}_i$, and $B^{(3)}_i$ for each $i = \overline{1,k}$ and also between the functions $C^{(1)}$ and $C^{(2)}$ and the sets $Q^{(1)}$ and $Q^{(2)}$. The prime means the transposition.

It is assumed that the initial positions of system (1.1) belong to a certain compact set $\mathcal{Y}$ in the game space Z; by the symbol $\mathcal{M}$, we denote a compact set in $\mathbb{R}^n$ that estimates from above the set of possible states of system (1.1) at the instant ϑ. Let us agree that

$$\|\gamma^{(1)} - \gamma^{(2)}\|_{\mathcal{M}} := \max_{x\in\mathcal{M}} |\gamma^{(1)}(x) - \gamma^{(2)}(x)|.$$

In the paper, we will prove the following assertion.

Theorem 1. *Let the conditions imposed on systems* (1.1) *and* (1.4) *and also on the functions* $V^{(2)}$ *and* $B^{(3)}$*, including Conditions* 1 and 2*, hold. Then for any* $\varepsilon > 0$*, there exist positive numbers* $r(\varepsilon)$ *and* $\Delta(\varepsilon)$ *such that for any strategy* U *of the first player that is a single-valued selection of the multivalued function* $\mathbf{U}^{r(\varepsilon)}$*, for any initial position* $(t_0,x_0) \in \mathcal{Y}$*, and for any step* $\Delta \le \Delta(\varepsilon)$ *of the discrete control scheme, the following estimate holds:*

$$\Gamma^{(1)}(t_0,x_0,U,\Delta) \le V^{(2)}(t_0,x_0) + \varepsilon + \lambda\chi(t_0,\vartheta) + \|\gamma^{(1)} - \gamma^{(2)}\|_{\mathcal{M}}. \tag{1.10}$$

Let us give some elucidation of the theorem. Performing the constructions in the framework of the approximating game, we know the value $V^{(2)}(t_0,x_0)$ of the function $V^{(2)}$ at the initial position (t_0,x_0). Therefore, on the right-hand side of estimate (1.10), we have $V^{(2)}(t_0,x_0)$. The distinction between the dynamics of the initial and approximation games and also the distinction of the function $B^{(3)}$ from the functions $B^{(1)}$ and $B^{(2)}$ are taken into account by the quantity $\chi(t_0,\vartheta)$. The summand $\|\gamma^{(1)} - \gamma^{(2)}\|_{\mathcal{M}}$ characterizes the difference between the cost functions. The switching sets $\Pi^{r(\varepsilon)}(i,t)$, $i = \overline{1,k}$, $t \in T$, for the multivalued function $\mathbf{U}^{r(\varepsilon)}$ are defined through constructions realized by using the functions $V^{(2)}$ and $B^{(3)}$.

On the whole, the right-hand side of (1.10) estimates the guarantee of the first player in game (1.1) when it uses (with step Δ) an arbitrary single-valued strategy U that is a selection of the multivalued function $\mathbf{U}^{r(\varepsilon)}$.

Since for any $i = \overline{1,k}$ and $t \in T$, the inclusion $\Pi(i,t) \subset \Pi^{r(\varepsilon)}(i,t)$ holds, it follows that outside the set $\Pi^{r(\varepsilon)}(i,t)$, the strategy U coincides with the function $\mathbf{U}^0$ defined by using the surfaces $\Pi(i,t)$. Let U^0 be a certain single-valued selection of the multivalued function $\mathbf{U}^0$. We obtain from what was said above that the action of the strategy U^0 performed with componentwise errors in the sets $\Pi^{r(\varepsilon)}(i,t)$, $i = \overline{1,k}$, $t \in T$, is also estimated by the right-hand side of (1.10). Therefore, we can speak about the stability of the strategy U^0 with respect to the inaccuracy of construction of the surfaces $\Pi(i,t)$ or with respect to the informational errors of measuring the state of the vector $y(t)$ with respect to the surfaces $\Pi(i,t)$.

Assume that the approximating game coincides with the initial game and the function $B^{(3)}$ coincides with the function $B^{(1)}$. Then $\chi(t_0,\vartheta) = 0$ and $\|\gamma^{(1)} - \gamma^{(2)}\|_{\mathcal{M}} = 0$. Moreover, as a u-stable function $V^{(2)}$, let us use the value function $\mathbf{\Gamma}^{(1)}$ of the initial state, and let Conditions 1 and 2 hold. As a result, we obtain

$$\Gamma^{(1)}(t_0, x_0, U, \Delta) \le V^{(1)}(t_0, x_0) + \varepsilon.$$

This means that any single-valued strategy U^0 defined by using the surfaces $\Pi(i,t)$ is a universal optimal strategy in game (1.1) and has the stability property.

Therefore, if the function $B^{(1)}$ and also the cost function $\gamma^{(1)}$ satisfy the Lipschitz condition, the switching surfaces $\Pi(i,t)$, $i = \overline{1,k}$, $t \in T$, are constructed in the framework of the initial game (1.1), and Conditions 1 and 2 hold, then as a universal optimal strategy U^*, we can take the strategy U^0.

The proof of Theorem 1 for $\beta > 0$ is given in Secs. 2–8. The case $\beta = 0$ is analyzed in Sec. 9 and uses the results of Secs. 2 and 3.

Theorem 1 gives no method for choosing the numbers $r(\varepsilon)$ and $\Delta(\varepsilon)$. Therefore, estimate (1.10) is not constructive.

In the case $k = 1$, i.e., when the control action u of the first player is scalar and satisfies the constraint $|u| \le \mu$, we can give an explicit estimate of the guarantee of the first player. In Sec. 10, we will prove the following assertion.

Theorem 2. *For $k = 1$, let the conditions imposed on systems (1.1) and (1.4) and also on the functions $V^{(2)}$ and $B^{(3)}$, including Condition 1, hold. Let $r \ge 0$ and $\Delta > 0$. Then for any single-valued strategy U of the first player that is a single-value selection of the multivalued function U^r and for any initial position $(t_0, x_0) \in \mathcal{Y}$, the following estimate holds:*

$$\Gamma^{(1)}(t_0, x_0, U, \Delta) \le V^{(2)}(t_0, x_0) + 2\lambda\sqrt{(2\sigma\mu\Delta + r)\beta\mu}(\vartheta - t_0) + 4\lambda\sigma\mu\Delta + \lambda r + \lambda\chi(t_0,\vartheta) + \|\gamma^{(1)} - \gamma^{(2)}\|_{\mathcal{M}}.$$

2. Main Lemma

2.1. Concept of nearness of a given vector to a set of other vectors. Let $\zeta > 0$. We say that a vector $a \in \mathbb{R}^n$ is ζ-close to a set of vectors $b_i \in \mathbb{R}^n$, $i = \overline{1,s}$, $s \ge 1$, if the module of the projection of the vector a on the orthogonal complement in $\mathbb{R}^n$ of the linear subspace spanned by the vectors b_i, $i = \overline{1,s}$, does not exceed the number ζ.

A vector $a \in \mathbb{R}^n$ is said to be ζ-small if $|a| \le \zeta$. Therefore, the ζ-smallness of the vector a means its ζ-closeness to the zero vector.

For $F \subset I$, $\zeta > 0$, and $t \in T$, by the symbol $\mathcal{H}(F,\zeta,t)$ we denote the set of all subscripts $j \in I \setminus F$ for each of which the vector $B_j^{(3)}(t)$ is ζ-close to the set of vectors $B_i^{(3)}(t)$, $i \in F$. If $F = \varnothing$, then $\mathcal{H}(F,\zeta,t)$ means the set of all $j \in I$ for each of which the vector $B_j^{(3)}(t)$ is ζ-small.

2.2. Formulation of the lemma and comments. For two compact sets X and Y in $\mathbb{R}^n$, let

$$\hat{d}(X,Y) := \max_{x\in X}\min_{y\in Y}|x-y|$$

be the Hausdorff deviation of the set X from the set Y. We set

$$G_v^{(i)}(t_*,t^*) := \bigcup_{v(\cdot)\in K^{(i)}} \int_{t_*}^{t^*} C^{(i)}(t)v(t)dt, \quad i=1,2.$$

The sets $G_v^{(1)}(t_*,t^*)$ and $G_v^{(2)}(t_*,t^*)$ are convex and compact. The following estimate holds:

$$\hat{d}(G_v^{(1)}(t_*,t^*), G_v^{(2)}(t_*,t^*)) \le \int_{t_*}^{t^*} \max_{|\ell|\le 1}\left[\max_{q\in Q^{(1)}} \ell' C^{(1)}(t)q - \max_{q\in Q^{(2)}} \ell' C^{(2)}(t)q\right] dt. \tag{2.1}$$

We denote by $G^{(1)}(t,t_*,x_*)$ the reachable set of system (1.1) at the instant t for the initial state x_* at the instant t_* and for all admissible open-loop controls $u(\cdot)$ and $v(\cdot)$ on the closed interval $[t_*,t]$.

Similarly, we denote by $G^{(2)}(t,t_*,x_*)$ the reachable set of system (1.4) at time t. We set

$$G^{(2)\natural}(t,t_*,x_*) := G^{(2)}(t,t_*,x_*) + B(2(t-t_*)\sigma\mu).$$

Here, $B(r)$ is the ball of radius r in $\mathbb{R}^n$.

The lemma formulated below is said to be the main one.

Lemma 2.1. *Let $(t_*,x_*)\in Z$, $\delta>0$, and $t_*+\delta\le\vartheta$. Fix a set $F\subset I$ and a number $\zeta>0$. Choose a certain set $H\subset\mathcal{H}(F,\zeta,t_*+\delta)$. Assume that for all $i\in I\setminus(F\cup H)$, $t\in[t_*,t_*+\delta]$, the following relations hold*:

$$G^{(1)}(t,t_*,x_*)\cap\Pi(i,t)=\varnothing, \quad G^{(2)\natural}(t,t_*,x_*)\cap\Pi(i,t)=\varnothing. \tag{2.2}$$

Further, assign a number $\omega\in[0,\delta]$, and let $y^{(1*)}(\cdot)$ be the motion of system (1.1) *with respect to admissible open-loop controls $u(\cdot)$ and $v(\cdot)$ starting from the point x_* at the instant t_*, and, moreover, for any $i\in I\setminus(F\cup H)$ and any $t\in[t_*+\omega,t_*+\delta]$, in the case $x_*\in\Pi_+(i,t_*)$ let the relation $u_i(t)=\mu_i$ hold, and in the case $x_*\in\Pi_-(i,t_*)$, let the relation $u_i(t)=-\mu_i$ hold. Then the following estimate holds*:

$$\begin{aligned}\mathcal{V}(F,t_*+\delta,y^{(1*)}(t_*+\delta)) &\le V^{(2)}(t_*,x_*) + \lambda\delta^2\sum_{i=1}^{k}\beta_i\mu_i \\ &+2\lambda\zeta\delta\sum_{i\in H}\mu_i + 2\lambda\omega\sum_{i\notin F\cup H}\sigma_i\mu_i + \lambda\chi(t_*,t_*+\delta).\end{aligned} \tag{2.3}$$

The meaning of this assertion consists in that $\mathcal{V}(F,t_*+\delta,y^{(1*)}(t_*+\delta))$ does not considerably increase as compared with the value of the function $V^{(2)}$ at the position (t_*,x_*), despite the fact that on the closed interval $[t_*,t_*+\delta]$, for subscripts $i\in F$, an arbitrary admissible control $u_i(\cdot)$ acts. For subscripts $j\in H$, the control $u_j(\cdot)$ is also arbitrary. In the case $i\notin F\cup H$, it is assumed that on the closed interval $[t_*+\omega,t_*\delta]$, a constant "regular" control of the first player, corresponding to the part of the space with respect to the surface $\Pi(i,t_*)$ in which the point x_* lies (i.e., in $\Pi_+(i,t_*)$ or $\Pi_-(i,t_*)$), acts. By condition, the motion $y^{(1*)}(t)$, emanating from the point $x_*\notin\Pi(i,t_*)$, $i\notin F\cup H$, at time t_* cannot enter $\Pi(i,t)$ for any $t\in[t_*,t_*+\delta]$.

2.3. Proof of the lemma. By the symbol $W_c^{(2)}$ we denote the level set (Lebesgue set) of function $V^{(2)}$ corresponding to the number c. The section at instant t is denoted by $W_c^{(2)}(t)$.

For the control $v(\cdot) \in K^{(1)}$ given in the condition of the lemma, in the set $G_v^{(1)}(t_*, t_* + \delta)$, we define the point

$$g := \int\limits_{t_*}^{t_*+\delta} C^{(1)}(t)v(t)dt.$$

Let $\overline{g}$ be the point of the set $G_v^{(2)}(t_*, t_* + \delta)$ nearest to it. Choose $\overline{v}(\cdot) \in K^{(2)}$ such that

$$\overline{g} = \int\limits_{t_*}^{t_*+\delta} C^{(2)}(t)\overline{v}(t)dt.$$

We set $c_* := V^{(2)}(t_*, x_*)$.

Using the u-stability of the function $V^{(2)}$, for the control $\overline{v}(\cdot)$, we find $\overline{u}(\cdot)$ such that for the motion $y^{(2*)}(t) := y^{(2*)}(t; t_*, x_*, \overline{u}(\cdot), \overline{v}(\cdot))$ emanating from the point x_* at the instant of time t_*, the following inclusion holds:

$$y^{(2*)}(t_* + \delta) \in W_{c_*}^{(2)}(t_* + \delta). \tag{2.4}$$

1. Denote

$$J_1 := \int\limits_{t_*}^{t_*+\delta} B^{(1)}(t)u(t)dt - \int\limits_{t_*}^{t_*+\delta} B^{(2)}(t)\overline{u}(t)dt, \quad J_2 := \int\limits_{t_*}^{t_*+\delta} C^{(1)}(t)v(t)dt - \int\limits_{t_*}^{t_*+\delta} C^{(2)}(t)\overline{v}(t)dt.$$

Then

$$y^{(1*)}(t_* + \delta) - y^{(2*)}(t_* + \delta) = J_1 + J_2.$$

We have

$$\begin{aligned} J_1 = &\sum_{i=1}^{k} \int\limits_{t_*}^{t_*+\delta} \Big(B_i^{(1)}(t) - B_i^{(3)}(t)\Big)u_i(t)\,dt - \sum_{i=1}^{k} \int\limits_{t_*}^{t_*+\delta} \Big(B_i^{(2)}(t) - B_i^{(3)}(t)\Big)\overline{u}_i(t)\,dt \\ &+ \sum_{i=1}^{k} \int\limits_{t_*}^{t_*+\delta} \Big(B_i^{(3)}(t) - B_i^{(3)}(t_*+\delta)\Big)\Big(u_i(t) - \overline{u}_i(t)\Big)\,dt \\ &+ \sum_{i=1}^{k} B_i^{(3)}(t_*+\delta) \int\limits_{t_*}^{t_*+\delta} \Big(u_i(t) - \overline{u}_i(t)\Big)dt, \end{aligned} \tag{2.5}$$

$$J_2 = g - \overline{g}. \tag{2.6}$$

2. For each $i \notin F \cup H$, on the closed interval $[t_*, t_* + \delta]$, we introduce an auxiliary constant control $u_{*i}(\cdot)$ equal to the constant value of the control $u_i(\cdot)$ on the closed interval $[t_* + \omega, t_* + \delta]$.

Denote

$$\hat{z} := y^{(2*)}(t_* + \delta) + \sum_{i \notin F \cup H} B_i^{(3)}(t_* + \delta) \int\limits_{t_*}^{t_*+\delta} (u_{*i}(t) - \overline{u}_i(t))dt.$$

Using estimate (2.1) and the assumption on the form of the control $u_i(\cdot)$ on the closed interval $[t_* + \omega, t_* + \delta]$, $i \in I \setminus (F \cup H)$, we prove that

$$V^{(2)}(t_* + \delta,\ \hat{z}) \le V^{(2)}(t_* + \delta, y^{(2*)}(t_* + \delta)). \tag{2.7}$$

Indeed, the assumption

$$G^{(1)}(t, t_*, x_*) \cap \Pi(i, t) = \varnothing, \quad i \notin F \cup H, \quad t \in [t_*, t_* + \delta],$$

means that for each $i \notin F \cup H$, any motion of system (1.1) either belongs to $\Pi_+(i,t)$ on the whole closed interval $[t_*, t_* + \delta]$ or belongs to $\Pi_-(i,t)$ also on this entire interval. The first case arises when $x_* \in \Pi_+(i,t_*)$, and the second arises when $x_* \in \Pi_-(i,t_*)$. We obtain from the condition imposed on the control action $u_i(t)$, $i \notin F \cup H$, that $u_{*i}(t) \equiv \mu_i$ in the first case and $u_{*i}(t) \equiv -\mu_i$ in the second.

We now turn to the condition

$$G^{(2)\natural}(t, t_*, x_*) \cap \Pi(i,t) = \varnothing, \quad i \notin F \cup H, \quad t \in [t_*, t_* + \delta].$$

This condition implies that for each $i \notin F \cup H$, the following inclusion holds in the case $x_* \in \Pi_+(i,t_*)$:

$$G^{(2)\natural}(t_* + \delta, t_*, x_*) \subset \Pi_+(i, t_* + \delta),$$

and in the case $x_* \in \Pi_-(i,t_*)$, the inclusion

$$G^{(2)\natural}(t_* + \delta, t_*, x_*) \subset \Pi_-(i, t_* + \delta)$$

holds.

Let us enumerate the subscripts $i \notin F \cup H$ in an arbitrary order: $i_1, i_2, \ldots, i_s$. Take the first subscript i_1. Assume that $x_* \in \Pi_+(i_1, t_*)$. Then

$$y^{(2*)}(t_* + \delta) \in \Pi_+(i_1, t_* + \delta),$$

$$z_{i_1} := y^{(2*)}(t_* + \delta) + B^{(3)}_{i_1}(t_* + \delta) \int\limits_{t_*}^{t_*+\delta} \Big(\mu_{i_1} - \overline{u}_{i_1}(t)\Big) dt \in G^{(2)\natural}\big(t_* + \delta, t_*, x_*\big) \subset \Pi_+(i_1, t_* + \delta).$$

Since $\mu_{i_1} \geq \overline{u}_{i_1}(t)$, $t \in [t_*, t_* + \delta]$, it follows from this that

$$V^{(2)}(t_* + \delta, z_{i_1}) \leq V^{(2)}(t_* + \delta, y^{(2*)}(t_* + \delta)).$$

We proceed analogously in the case $x_* \in \Pi_-(i_1, t_*)$, but now we use the inequality $-\mu_{i_1} \leq \overline{u}_{i_1}(t)$, $t \in [t_*, t_* + \delta]$.

Let us pass to the second subscript i_2. Assume that $x_* \in \Pi_+(i_2, t_*)$. Then

$$z_{i_1} \in \Pi_+(i_2, t_* + \delta),$$

$$z_{i_2} := z_{i_1} + B^{(3)}_{i_2}(t_* + \delta) \int\limits_{t_*}^{t_*+\delta} (\mu_{i_2} - \overline{u}_{i_2}(t)) dt \in G^{(2)\natural}(t_* + \delta, t_*, x_*) \subset \Pi_+(i_2, t_* + \delta).$$

Since $\mu_{i_2} \geq \overline{u}_{i_2}(t)$, $t \in [t_*, t_* + \delta]$, it follows from this that

$$V^{(2)}(t_* + \delta, z_{i_2}) \leq V^{(2)}(t_* + \delta, z_{i_1}).$$

We proceed analogously in the case $x_* \in \Pi_-(i_2, t_*)$.

Continuing this process sequentially up to the last subscript i_s, we arrive at the following chain of inequalities:

$$V^{(2)}(t_* + \delta, z_{i_s}) \leq V^{(2)}(t_* + \delta, z_{i_{s-1}}) \leq \ldots V^{(2)}(t_* + \delta, z_{i_1}) \leq V^{(2)}(t_* + \delta, y^{(2*)}(t_* + \delta)).$$

Therefore,

$$V^{(2)}(t_* + \delta, \hat{z}) = V^{(2)}(t_* + \delta, z_{i_s}) \leq V^{(2)}(t_* + \delta, y^{(2*)}(t_* + \delta)).$$

Inequality (2.7) is proved.

By (2.4), we obtain from (2.7) the inclusion

$$\hat{z} \in W^{(2)}_{c_*}(t_* + \delta). \tag{2.8}$$

3. Using the notation introduced by formulas (2.5) and (2.6), we have

$$y^{(1*)}(t_* + \delta) - \hat{z} = y^{(1*)}(t_* + \delta) - y^{(2*)}(t_* + \delta) - \sum_{i \notin F \cup H} B^{(3)}_i(t_* + \delta) \int\limits_{t_*}^{t_*+\delta} (u_{*i}(t) - \overline{u}_i(t)) dt$$

$$= J_1 + J_2 - \sum_{i \notin F \cup H} B_i^{(3)}(t_* + \delta) \int\limits_{t_*}^{t_*+\delta} (u_{*i}(t) - \overline{u}_i(t))dt.$$

By the symbol π we denote the operator of orthogonal projection of the space $\mathbb{R}^n$ on the subspace orthogonal to the subspace spanned by the vectors $B_i^{(3)}(t_* + \delta)$, $i \in F$.

Now we take into account that the modules of the controls $u_i(t)$ and $\overline{u}_i(t)$ are bounded by the number μ_i and each of the functions $B_i^{(3)}$ satisfies the Lipshitz condition with constant β_i. Also, we take into account that $\pi B_i^{(3)}(t_* + \delta) = 0$ for $i \in F$ and any vector $B_j^{(3)}(t_* + \delta)$, $j \in H$, is ζ-close to the set of vectors $B_i^{(3)}(t_* + \delta)$, $i \in F$.

We obtain

$$\left| \pi J_1 - \pi \sum_{i \notin F \cup H} B_i^{(3)}(t_* + \delta) \int\limits_{t_*}^{t_*+\delta} (u_{*i}(t) - \overline{u}_i(t))dt \right|$$

$$\leq \sum_{i=1}^{k} \mu_i \int\limits_{t_*}^{t_*+\delta} \Big(\big|B_i^{(1)}(t) - B_i^{(3)}(t)\big| + \big|B_i^{(2)}(t) - B_i^{(3)}(t)\big| \Big) dt$$

$$+ \delta^2 \sum_{i=1}^{k} \beta_i \mu_i + 2\zeta\delta \sum_{i \in H} \mu_i + \left| \pi \sum_{i \notin F \cup H} B_i^{(3)}(t_* + \delta) \int\limits_{t_*}^{t_*+\delta} (u_i(t) - u_{*i}(t))dt \right|.$$

By (2.6) and (2.1), we have

$$|\pi J_2| = |\pi(g - \overline{g})| \leq |g - \overline{g}| \leq \int\limits_{t_*}^{t_*+\delta} \max_{|\ell| \leq 1} \left[\max_{q \in Q^{(1)}} \ell' C^{(1)}(t) q - \max_{q \in Q^{(2)}} \ell' C^{(2)}(t) q \right] dt.$$

Taking into account the definition of the functions $u_{*i}(t)$ on $[t_*, t_* + \delta]$ and also the inequalities

$$|B_i^{(3)}(t_* + \delta)| \leq \sigma_i, \quad i \notin F \cup H,$$

we obtain

$$\left| \pi \sum_{i \notin F \cup H} B_i^{(3)}(t_* + \delta) \int\limits_{t_*}^{t_*+\delta} \big(u_i(t) - u_{*i}(t)\big) dt \right|$$

$$= \left| \pi \sum_{i \notin F \cup H} B_i^{(3)}(t_* + \delta) \int\limits_{t_*}^{t_*+\omega} \big(u_i(t) - u_{*i}(t)\big) dt \right| \leq 2\omega \sum_{i \notin F \cup H} \sigma_i \mu_i.$$

Finally, we arrive at the inequality

$$|\pi y^{(1*)}(t_* + \delta) - \pi \hat{z}| \leq \delta^2 \sum_{i=1}^{k} \beta_i \mu_i + 2\zeta\delta \sum_{i \in H} \mu_i + 2\omega \sum_{i \notin F \cup H} \sigma_i \mu_i + \chi(t_*, t_* + \delta). \tag{2.9}$$

Let $\tilde{x}$ be the point nearest to the set $W_{c_*}^{(2)}(t_* + \delta)$ on the plane $\mathcal{A}(F, t_* + \delta, y^{(1*)}(t_* + \delta))$. It follows from (2.8) and the definition of the operator π that

$$\hat{d}(\{\tilde{x}\}, W_{c_*}^{(2)}(t_* + \delta)) \leq |\tilde{x} - \hat{z}| \leq |\pi\tilde{x} - \pi\hat{z}| = |\pi y^{(1*)}(t_* + \delta) - \pi\hat{z}|.$$

Therefore,

$$V^{(2)}(t_* + \delta, \tilde{x}) \leq c_* + \lambda |\pi y^{(1*)}(t_* + \delta) - \pi\hat{z}| = V^{(2)}(t_*, x_*) + \lambda |\pi y^{(1*)}(t_* + \delta) - \pi\hat{z}|.$$

With account for (2.9), the required inequality (2.3) follows from the inequality

$$\mathcal{V}(F, t_* + \delta, y^{(1*)}(t_* + \delta)) \le V^{(2)}(t_* + \delta, \tilde{x}).$$

2.4. Remarks. 1. Consider the degenerate case where $F = \varnothing$. In this case, the set H consists of subscripts $i = \overline{1, k}$ for each of which $|B_i^{(3)}(t_* + \delta)| \le \zeta$. Estimate (2.3) preserves its form, but

$$\mathcal{V}(F, t_* + \delta, y^{(1*)}(t_* + \delta)) = V^{(2)}(t_* + \delta, y^{(1*)}(t_* + \delta)).$$

Therefore, in the degenerate case $F = \varnothing$, we have

$$\begin{aligned} V^{(2)}(t_* + \delta, y^{(1*)}(t_* + \delta)) &\le V^{(2)}(t_*, x_*) + \lambda\delta^2 \sum_{i=1}^{k} \beta_i \mu_i \\ &+ 2\lambda\zeta\delta \sum_{i \in H} \mu_i + 2\lambda\omega \sum_{i \notin H} \sigma_i \mu_i + \lambda\chi(t_*, t_* + \delta). \end{aligned} \tag{2.10}$$

2. Let the set F be such that the number of linearly independent vectors $B_i^{(3)}(t_* + \delta)$, $i \in F$, is equal to n, i.e., coincides with the dimension of the space $\mathbb{R}^n$. Then

$$\mathcal{V}(F, t_* + \delta, y^{(1*)}(t_* + \delta)) = \min_{z \in \mathbb{R}^n} V^{(2)}(t_* + \delta, z) \le \min_{z \in \mathbb{R}^n} V^{(2)}(t_*, z) \le V^{(2)}(t_*, x_*).$$

Therefore,

$$\mathcal{V}(F, t_* + \delta, y^{(1*)}(t_* + \delta)) \le V^{(2)}(t_*, x_*).$$

Moreover, the controls $u_i(\cdot), i \notin F$, on the closed interval $[t_*, t_* + \delta]$, as well as the controls $u_i(\cdot)$, $i \in F$, can be arbitrary.

2.5. Estimate of the variation of the function $V^{(2)}$ in a particular case. Let us formulate an assertion that follows from estimate (2.10).

Proposition 2.1. *Let $(t_*, x_*) \in Z, t^* \in (t_*, \vartheta]$. Assign a number $\zeta > 0$ and choose a set of subscripts $H \subset I$ so that for any $j \in H$ and any $t \in [t_*, t^*]$, the inequality $|B_j^{(3)}(t)| \le \zeta$ holds. Let $0 \le \omega \le t^* - t_*$, and along the motion $y^{(1*)}(\cdot)$ emanating from the point x_* at the instant t_*, for any $i \in I \setminus H$, either $y^{(1*)}(t) \in \Pi_+(i, t)$ on the closed interval $[t_*, t^*]$ and, moreover, $u_i(t) = \mu_i$ on $[t_* + \omega, t^*]$ or $y^{(1*)}(t) \in \Pi_-(i, t)$ on the closed interval $[t_*, t^*]$ and, moreover, $u_i(t) = -\mu_i$ on $[t_* + \omega, t^*]$. Then the following estimate holds for any $t \in [t_*, t^*]$:*

$$V^{(2)}(t, y^{(1*)}(t)) \le V^{(2)}(t_*, x_*) + 2\lambda\zeta(t - t_*) \sum_{i \in H} \mu_i + 2\lambda\omega \sum_{i \notin H} \sigma_i \mu_i + \lambda\chi(t_*, t). \tag{2.11}$$

Proof. Divide the closed interval $[t_*, t^*]$ by instants t_s, $s = \overline{1, e}$, $t_1 = t_*$, $t_e = t^*$, with the step δ so that for any closed interval $[t_s, t_{s+1}]$, $s = 1, 2, \dots, e - 1$, of the obtained partition and for $t \in [t_s, t_{s+1}]$, $i \notin H$, the following conditions hold:

$$G^{(1)}(t, t_s, y^{(1*)}(t_s)) \cap \Pi(i, t) = \varnothing, \quad G^{(2)\natural}(t, t_s, y^{(1*)}(t_s)) \cap \Pi(i, t) = \varnothing.$$

This can be done by using the assumption imposed on the location of $y^{(1*)}(t)$ with respect to the surfaces $\Pi(i, t)$, $i \notin H$.

By (2.10), for each s such that $t_s > t_* + \omega$, we have the relation

$$V^{(2)}(t_{s+1}, y^{(1*)}(t_{s+1})) \le V^{(2)}(t_s, y^{(1*)}(t_s)) + \lambda\delta^2 \sum_{i=1}^{n} \beta_i \mu_i + 2\lambda\zeta\delta \sum_{i \in H} \mu_i + \lambda\chi(t_s, t_{s+1}). \tag{2.12}$$

For s such that $t_s \in [t_*, t_* + \omega]$, by (2.10) we obtain

$$V^{(2)}(t_{s+1}, y^{(1*)}(t_{s+1})) \le V^{(2)}(t_s, y^{(1*)}(t_s)) + \lambda\delta^2 \sum_{i=1}^{n} \beta_i \mu_i + 2\lambda\zeta\delta \sum_{i \in H} \mu_i + 2\lambda\delta \sum_{i \notin H} \sigma_i \mu_i + \lambda\chi(t_s, t_{s+1}). \tag{2.13}$$

Sequentially applying estimates (2.12) and (2.13) for $s = 1, 2, \ldots, e-1$, we arrive at the inequality

$$V^{(2)}(t^*, y^{(1*)}(t^*)) \le V^{(2)}(t_*, x_*) + \lambda(t^* - t_*)\delta \sum_{i=1}^{k} \beta_i \mu_i + 2\lambda\zeta(t^* - t_*) \sum_{i \in H} \mu_i + 2\lambda(\omega + \delta) \sum_{i \notin H} \sigma_i \mu_i + \lambda\chi(t_*, t^*).$$

Passing to the limit as $\delta \to 0$, we obtain estimate (2.11). □

Let us apply the assertion just proved to the case where all control actions $u_i(t)$, $i = \overline{1,k}$, are arbitrary on $[t_*, t^*]$ and a certain smallness of the vectors $B_i^{(3)}(t)$, $i = \overline{1,k}$, $t \in [t_*, t^*]$, specially specified, is not presupposed. As the characteristic of smallness, we take

$$\zeta = \sigma = \max_{\substack{i=\overline{1,k} \\ t \in T}} |B_i^{(3)}(t)|.$$

The following assertion holds.

Proposition 2.2. *Let $(t_*, x_*) \in Z$, $t^* \in (t_*, \vartheta]$. Let the motion $y^{(1*)}(\cdot)$ on $[t_*, t^*]$ emanating from the point x_* at the instant t_* be generated by arbitrary admissible open-loop controls $u(\cdot)$ and $v(\cdot)$ of the first and second players. Then the following estimate holds for any $t \in [t_*, t^*]$:*

$$V^{(2)}(t, y^{(1*)}(t)) \le V^{(2)}(t_*, x_*) + 2\lambda\sigma\mu(t - t_*) + \lambda\chi(t_*, t). \tag{2.14}$$

3. Sets $\Pi^c(F, t)$

3.1. Definition of the sets $\Pi^c(F, t)$. The sets $\Pi(i, t)$ introduced by formulas (1.5)–(1.7) have the upper semicontinuity property in the argument t. However, there cannot be lower semicontinuity in t even in the case where the vector $B_i^{(3)}(t)$ using which we construct the set $\Pi(i, t)$ does not vanish on the interval T.

The sets $\Pi^r(i, t)$ are a geometric r-extension of the sets $\Pi(i, t)$, and, therefore, they are also upper semicontinuous in t, but there can be no lower semicontinuity. Therefore, unfortunately, we cannot speak about the continuous variation of sets $\Pi^r(i, t)$ in the argument t.

In this connection, we consider one more variant of extension of the sets $\Pi(i, t)$, but using only the quantity c, which, in contrast to r, means not the distance in the direction of the vector $B_i^{(3)}(t)$ or in the opposite direction but the overfall of values of the function $V^{(2)}$. Precisely, for all $i = \overline{1,k}$, $t \in T$, and $c \ge 0$, we set

$$\Pi^c(i, t) := \{x \in \mathbb{R}^n : V^{(2)}(t, x) - \mathcal{V}(i, t, x) \le c\}. \tag{3.1}$$

For $c = 0$, we have $\Pi^c(i, t) = \Pi(i, t)$.

We agree to call the set $\Pi^c(i, t)$ a c-neighborhood of the surface $\Pi(i, t)$ and distinguish it from the set $\Pi^r(i, t)$, i.e., from the r-neighborhood of the surface $\Pi(i, t)$.

Also, we will need the sets

$$\Pi^c(F, t) := \{x \in \mathbb{R}^n : V^{(2)}(t, x) - \mathcal{V}(F, t, x) \le c\}, \tag{3.2}$$

which will be considered for all $F \subset I$, $t \in T$ and $c \ge 0$. Formula (3.1) is a particular case of (3.2), where F consists of a single element.

If $c_* < c^*$, then

$$\Pi^{c_*}(F, t) \subset \operatorname{int} \Pi^{c^*}(F, t).$$

Here, int is the interior of a set.

Below we will show that if for any t from a certain closed interval $\mathcal{T} \subset T$, the vectors $B_i^{(3)}(t)$, $i \in F$, are linearly independent, then the function $(t, x) \mapsto \mathcal{V}(F, t, x)$ is continuous on the set $\mathcal{T} \times \mathbb{R}^n$. We will prove the upper semicontinuity property of the sets $\Pi^c(F, t)$ for $c \ge 0$, $t \in T$. We will prove that if on a closed set $\mathcal{T} \subset T$, the vector $B_i^{(3)}(t)$ does not vanish, then the multivalued function $(c,t) \mapsto \Pi^c(i,t)$ is continuous for $c > 0$, $t \in \mathcal{T}$.

3.2. Continuity of the functions $\mathcal{V}(F,\cdot,\cdot)$. In this subsection, we will prove the following assertion.

Lemma 3.1. *For any set $F \subset I$, the upper semicontinuity property of the function $(t,x) \mapsto \mathcal{V}(F,t,x)$ holds on the set Z. If, for a certain closed interval $\mathcal{T} \subset T$, the vectors $B_i^{(3)}(t)$, $i \in F$, are linearly independent for any $t \in \mathcal{T}$, then the function $(t,x) \mapsto \mathcal{V}(F,t,x)$ also has the lower semicontinuity property on the set $\mathcal{T} \times \mathbb{R}^n$.*

Proof. Preparatorily, as an obvious fact, we note the lower semicontinuity property of the function $(t,x) \mapsto \mathcal{A}(F,t,x)$ on the set $T \times \mathbb{R}^n$ and its upper semicontinuity on the set $\mathcal{T} \times \mathbb{R}^n$ under the additional assumption on the linear independence of the vectors $B_i^{(3)}(t)$, $i \in F$, for each $t \in \mathcal{T}$.

1. In the set Z, we consider an arbitrary sequence $(t_n, x_n) \to (t^*, x^*)$. Let us show that

$$\varlimsup_{n\to\infty} \mathcal{V}(F,t_n,x_n) \leq \mathcal{V}(F,t^*,x^*). \tag{3.3}$$

Let a point z^* on the plane $\mathcal{A}(F,t^*,x^*)$ be such that $V^{(2)}(t^*,z^*) = \mathcal{V}(F,t^*,x^*)$. The lower semicontinuity of the function $\mathcal{A}(F,\cdot,\cdot)$ implies that for each $n = 1,2,\ldots$, there exists $z_n \in \mathcal{A}(F,t_n,x_n)$ such that $z_n \to z^*$. For example, we can take

$$z_n = x_n + \sum_{i\in F} b_i^* B_i^{(3)}(t_n),$$

where the coefficients b_i^*, $i \in F$, satisfy the relation

$$z_* = x_* + \sum_{i\in F} b_i^* B_i^{(3)}(t^*).$$

Since $\mathcal{V}(F,t_n,x_n) \leq V^{(2)}(t_n,z_n)$, it follows that

$$\varlimsup_{n\to\infty} \mathcal{V}(F,t_n,x_n) \leq \lim_{n\to\infty} V^{(2)}(t_n,z_n) = V^{(2)}(t^*,z^*) = \mathcal{V}(F,t^*,x^*).$$

Therefore, relation (3.3) expressing the upper semicontinuity of the function $\mathcal{V}(F,\cdot,\cdot)$ is proved.

Now let us assume that the vectors $B_i^{(3)}(t)$, $i \in F$, are linearly independent for any t from the closed interval $\mathcal{T} \subset T$. Choose an arbitrary sequence $(t_n,x_n) \to (t^*,x^*)$, $(t_n,x_n) \in \mathcal{T} \times \mathbb{R}^n$. Let us prove the inequality

$$\varliminf_{n\to\infty} \mathcal{V}(F,t_n,x_n) \geq \mathcal{V}(F,t^*,x^*). \tag{3.4}$$

For each $n = 1,2,\ldots$, let a point z_n on the plane $\mathcal{A}(F,t_n,x_n)$ be such that $V^{(2)}(t_n,z_n) = \mathcal{V}(F,t_n,x_n)$.

A. We first prove the boundedness of the sequence z_n. The boundedness property is a consequence of an infinite growth $|\gamma^{(2)}(x)| \to \infty$ of the cost function $\gamma^{(2)}$ as $|x| \to \infty$.

We have $V^{(2)}(t_n,x_n) \to V^{(2)}(t^*,x^*)$. Given a function $\kappa > 0$, we choose a number N such that the following inequality holds for $n \geq N$:

$$V^{(2)}(t_n,x_n) \leq V^{(2)}(t^*,x^*) + \kappa.$$

Consider the level set $M_{\kappa^*}^{(2)} := \{x \in \mathbb{R}^n : \gamma^{(2)}(x) \leq \kappa^*\}$ of the function $\gamma^{(2)}$ corresponding to the number $\kappa^* := V^{(2)}(t^*,x^*) + \kappa$. The set $M_{\kappa^*}^{(2)}$ is bounded. But then, uniformly in $t \in T$, the set $W_{\kappa^*}^{(2)}(t)$ is bounded. Since $z_n \in W_{\kappa^*}^{(2)}(t_n)$, it follows that the sequence z_n is bounded.

B. From the sequence z_n, we extract a convergent subsequence z_k realizing the lower limit $\varliminf\limits_{n\to\infty} V^{(2)}(t_n,z_n)$. Let $z_k \to \overline{z}$. Then

$$\varliminf_{n\to\infty} \mathcal{V}(F,t_n,x_n) = \varliminf_{n\to\infty} V^{(2)}(t_n,z_n) = \lim_{k\to\infty} V^{(2)}(t_k,z_k) = V^{(2)}(t^*,\overline{z}). \tag{3.5}$$

By the assumption of the linear independence of the vectors $B_i^{(3)}(t)$, $i \in F$, for each $t \in \mathcal{T}$, we have the upper semicontinuity property of the function $\mathcal{A}(F,\cdot,\cdot)$ on the set $\mathcal{T} \times \mathbb{R}^n$. Therefore, the conditions $z_k \in \mathcal{A}(F, t_k, x_k)$ and $z_k \to \overline{z}$ imply the inclusion $\overline{z} \in \mathcal{A}(F, t^*, x^*)$. Hence

$$V^{(2)}(t^*, \overline{z}) \geq \mathcal{V}(F, t^*, x^*).$$

As a result, taking into account (3.5), we have

$$\varliminf_{n\to\infty} \mathcal{V}(F, t_n, x_n) = V^{(2)}(t^*, \overline{z}) \geq \mathcal{V}(F, t^*, x^*).$$

Therefore, relation (3.4), which means the lower semicontinuity of the function $\mathcal{V}(F,\cdot,\cdot)$, is proved. □

3.3. Upper semicontinuity of the mapping $(c,t) \mapsto \Pi^c(F,t)$. Continuity of the mapping $(c,t) \mapsto \Pi^c(i,t)$.

Proposition 3.1. *For any set $F \subset I$, the mapping $(c,t) \mapsto \Pi^c(F,t)$ is upper semicontinuous on the set $c \geq 0$, $t \in T$.*

Proof. Fix arbitrary $c^* \geq 0$ and $t^* \in T$ and consider arbitrary sequences $c_n \to c^*$ and $t_n \to t^*$. For each $n = 1, 2, \ldots$, choose $z_n \in \Pi^{c_n}(F, t_n)$ and assume that $z_n \to z^*$. Let us show that $z^* \in \Pi^{c^*}(F, t^*)$. This means the upper semicontinuiuty.

We have

$$V^{(2)}(t_n, z_n) - \mathcal{V}(F, t_n, z_n) \leq c_n \quad \Rightarrow \quad V^{(2)}(t_n, z_n) \leq c_n + \mathcal{V}(F, t_n, z_n).$$

Taking into account the upper semicontinuity of the function $\mathcal{V}(F,\cdot,\cdot)$, we obtain from this that

$$V^{(2)}(t^*, z^*) = \lim_{n\to\infty} V^{(2)}(t_n, z_n) \leq \lim_{n\to\infty} c_n + \varlimsup_{n\to\infty} \mathcal{V}(F, t_n, z_n) \leq c^* + \mathcal{V}(F, t^*, z^*).$$

Therefore,

$$V^{(2)}(t^*, z^*) - \mathcal{V}(F, t^*, z^*) \leq c^*.$$

Hence $z^* \in \Pi^{c^*}(F, t^*)$. □

Lemma 3.2. *Let $i \in I$, and on a certain closed interval $\mathcal{T} \subset T$, let the vector $B_i^{(3)}(t)$ do not vanish. Then on the set $c > 0$, $t \in \mathcal{T}$, the mapping $(c,t) \mapsto \Pi^c(i,t)$ is continuous.*

Proof. Taking into account Proposition 3.1, we see that it suffices to prove the lower semicontinuity of the mapping $(c,t) \mapsto \Pi^c(i,t)$ on the set $c > 0$, $t \in \mathcal{T}$.

Let $c^* > 0$, $t^* \in \mathcal{T}$, and certain sequences $c_n \to c^*$ and $t_n \to t^*$, where $t_n \in \mathcal{T}$, be given. Take an arbitrary $z^* \in \Pi^{c^*}(i, t^*)$. Let us show the possibility of choosing $z_n \in \Pi^{c_n}(i, t_n)$ such that $z_n \to z^*$. This means the lower semicontinuity.

Define a point $y^* \in \mathcal{A}(i, t^*, z^*)$ such that $V^{(2)}(t^*, y^*) = \mathcal{V}(i, t^*, z^*)$. We have $V^{(2)}(t^*, y^*) \leq V^{(2)}(t^*, z^*)$.

1. Consider the case where $V^{(2)}(t^*, y^*) < V^{(2)}(t^*, z^*)$. Then $y^* \neq z^*$. Note that for any point z on the line passing through y^* and z^*, the relation $\mathcal{V}(i, t^*, z) = \mathcal{V}(i, t^*, z^*)$ holds.

Fix a positive $\kappa < c^*$. Using the continuity of the functions $\mathcal{V}(i,\cdot,y^*)$ and $V^{(2)}(\cdot, y^*)$, we choose a number $\widetilde{N}$ such that the following inequalities hold for any $n \geq \widetilde{N}$:

$$\mathcal{V}(i, t_n, y^*) \geq \mathcal{V}(i, t^*, y^*) - \frac{\kappa}{2}, \quad V^{(2)}(t_n, y^*) \leq V^{(2)}(t^*, y^*) + \frac{\kappa}{2}, \quad c_n \geq \kappa.$$

Then for $n \geq \widetilde{N}$, we have

$$V^{(2)}(t_n, y^*) - \mathcal{V}(i, t_n, y^*) \leq V^{(2)}(t^*, y^*) + \frac{\kappa}{2} - \mathcal{V}(i, t^*, y^*) + \frac{\kappa}{2} = \kappa \leq c_n.$$

Therefore,

$$V^{(2)}(t_n, y^*) - \mathcal{V}(i, t_n, y^*) \leq c_n, \quad n \geq \widetilde{N}. \tag{3.6}$$

Further, we assume that $n \geq \widetilde{N}$.

On the closed interval $[y^*, z^*]$, we consider points z for which

$$V^{(2)}(t_n, z) - \mathcal{V}(i, t_n, z) \le c_n. \tag{3.7}$$

By (3.6), there exists at least one such point $z = y^*$. Using the continuity of the functions $V^{(2)}(t_n, \cdot)$ $\mathcal{V}(i, t_n, \cdot)$, among the points $z \in [y^*, z^*]$ satisfying (3.7) we choose a point nearest to z^* and denote it by z_n. Note that if $z_n \neq z^*$, then for such points, we have the equality in (3.7).

Let us show that $z_n \to z^*$. Assume the contrary, i.e., let there exist a subsequence $z_k \to \hat{z}$, $\hat{z} \neq z^*$. We assume that $z_k \neq z^*$ for any k. Then using the subscript k instead of the subscript n and the symbol z_k instead of z in (3.7), we obtain the equality. Precisely,

$$V^{(2)}(t_k, z_k) - \mathcal{V}(i, t_k, z_k) = c_k. \tag{3.8}$$

We set

$$\eta := V^{(2)}(t^*, z^*) - V^{(2)}(t^*, \hat{z}).$$

Taking into account Condition 1 and the inequality $V^{(2)}(t^*, y^*) < V^{(2)}(t^*, z^*)$, we have $\eta > 0$. Choose $\overline{N}$ such that the following inequalities hold for any $k \ge \overline{N}$:

$$\begin{gathered} \mathcal{V}(i, t_k, z_k) \ge \mathcal{V}(i, t^*, \hat{z}) - \frac{\eta}{4} = \mathcal{V}(i, t^*, z^*) - \frac{\eta}{4}, \\ V^{(2)}(t_k, z_k) \le V^{(2)}(t_*, \hat{z}) + \frac{\eta}{4}, \quad c_k \ge c^* - \frac{\eta}{4}. \end{gathered} \tag{3.9}$$

Using (3.8) and (3.9), we obtain

$$\begin{gathered} V^{(2)}(t^*, z^*) - \mathcal{V}(i, t^*, z^*) = V^{(2)}(t^*, z^*) - V^{(2)}(t_*, \hat{z}) + V^{(2)}(t_*, \hat{z}) \\ -V^{(2)}(t_k, z_k) + V^{(2)}(t_k, z_k) - \mathcal{V}(i, t_k, z_k) + \mathcal{V}(i, t_k, z_k) - \mathcal{V}(i, t^*, z^*) \\ \ge \eta - \frac{\eta}{4} + c_k - \frac{\eta}{4} \ge \eta - \frac{\eta}{4} + c^* - \frac{\eta}{4} - \frac{\eta}{4} = c^* + \frac{\eta}{4}. \end{gathered}$$

The inequality

$$V^{(2)}(t^*, z^*) - \mathcal{V}(i, t^*, z^*) \ge c^* + \frac{\eta}{4}$$

contradicts the inclusion $z^* \in \Pi^{c^*}(i, t^*)$.

Therefore, we have proved that $z_n \to z^*$. Moreover, $z_n \in \Pi^{c_n}(i, t_n)$.

2. Consider the case where $V^{(2)}(t^*, y^*) = V^{(2)}(t^*, z^*)$. We have

$$V^{(2)}(t^*, z^*) - \mathcal{V}(i, t^*, z^*) = 0.$$

Using the continuity of the functions $V^{(2)}(\cdot, z^*)$ and $\mathcal{V}(i, \cdot, z^*)$, we choose $\widehat{N}$ so that the following inequalities hold for any $n \ge \widehat{N}$:

$$V^{(2)}(t_n, z^*) - \mathcal{V}(i, t_n, z^*) \le \frac{c^*}{2}, \quad c_n \ge \frac{c^*}{2}.$$

Then

$$V^{(2)}(t_n, z^*) - \mathcal{V}(i, t_n, z^*) \le c_n.$$

The latter inequality means that $z^* \in \Pi^{c_n}(i, t_n)$ for $n \ge \widehat{N}$. Therefore, in the desired sequence, we can take $z_n = z^*$ for $n \ge \widehat{N}$. □

3.4. Assertion on the zero distance between the part of the set $\Pi^{c_*}(\mathfrak{F}_1, t)$ located outside the set $\Pi^{\overline{c}}(\mathfrak{F}_1 \cup \mathfrak{F}_2, t)$ and the set $\Pi^{c_*}(\mathfrak{F}_2, t)$. Let us formulate a consequence of Lemma 3.1 and Proposition 3.1. In this case, the symbol d means the distance between two sets:

$$d(A, B) := \inf \big\{ |a - b| : a \in A, \ b \in B \big\}.$$

Lemma 3.3. *Let $F \subset I$ be a certain set consisting of more than one element and let $\mathcal{T}$ be a closed interval in T. Assume that the vectors $B_i^{(3)}(t)$, $i \in F$, are linearly independent for any $t \in \mathcal{T}$. Fix a number $\overline{c} > 0$. Decompose the set F into disjoint subsets $\mathfrak{F}_1$ and $\mathfrak{F}_2$. Then for any bounded set $\mathcal{X} \subset \mathbb{R}^n$, there exist positive numbers c_* and e_* such that*

$$d\Bigg(\Big(\Pi^{c_*}(\mathfrak{F}_1, t) \cap \mathcal{X}\Big) \setminus \operatorname{int} \Pi^{\overline{c}}(F, t),\ \Pi^{c_*}(\mathfrak{F}_2, t)\Bigg) \geq e_*$$

for all $t \in \mathcal{T}$.

Proof. Assuming the contrary, we choose subsequences of positive numbers $c_n \to 0$ and $e_n \to 0$ and then sequences $t_n \in \mathcal{T}$ of instants of time and points

$$z_{1n} \in (\Pi^{c_n}(\mathfrak{F}_1, t_n) \cap \mathcal{X}) \setminus \operatorname{int} \Pi^{\overline{c}}(F, t_n), \quad z_{2n} \in \Pi^{c_n}(\mathfrak{F}_2, t_n)$$

so that $|z_{1n} - z_{2n}| \leq e_n$. Using the boundedness of the sets $\mathcal{T}$ and $\mathcal{X}$, we extract convergent subsequences $t_k \to t_*$ and $z_{1k} \to z_*$. For the corresponding subsequence z_{2k} of points z_{2n}, we obtain $z_{2k} \to z_*$.

By the upper semicontinuity of the mappings $(c, t) \mapsto \Pi^c(\mathfrak{F}_1, t)$, $(c, t) \mapsto \Pi^c(\mathfrak{F}_2, t)$, we have

$$z_* \in \Pi(\mathfrak{F}_1, t_*), \quad z_* \in \Pi(\mathfrak{F}_2, t_*).$$

Then, using Condition 2, we conclude that $z_* \in \Pi(F, t_*)$, and, therefore,

$$V^{(2)}(t_*, z_*) = \mathcal{V}(F, t_*, z_*). \tag{3.10}$$

Using the continuity of the functions $V^{(2)}$ and $\mathcal{V}(F, \cdot, \cdot)$ on $\mathcal{T} \times \mathbb{R}^n$, we choose a number N such that the following inequalities hold for $k \geq N$:

$$|V^{(2)}(t_k, z_{1k}) - V^{(2)}(t_*, z_*)| \leq \frac{\overline{c}}{4}, \tag{3.11}$$

$$|\mathcal{V}(F, t_k, z_{1k}) - \mathcal{V}(F, t_*, z_*)| \leq \frac{\overline{c}}{4}. \tag{3.12}$$

Based on relations (3.10)–(3.12), for $k \geq N$ we obtain

$$\begin{aligned} &V^{(2)}(t_k, z_{1k}) - \mathcal{V}(F, t_k, z_{1k}) \\ &= V^{(2)}(t_k, z_{1k}) - V^{(2)}(t_*, z_*) + \mathcal{V}(F, t_*, z_*) - \mathcal{V}(F, t_k, z_{1k}) \\ &\leq |V^{(2)}(t_k, z_{1k}) - V^{(2)}(t_*, z_*)| + |\mathcal{V}(F, t_*, z_*) - \mathcal{V}(F, t_k, z_{1k})| \leq 2 \cdot \frac{\overline{c}}{4} = \frac{\overline{c}}{2}. \end{aligned}$$

The inequality

$$V^{(2)}(t_k, z_{1k}) - \mathcal{V}(F, t_k, z_{1k}) \leq \frac{\overline{c}}{2}, \quad k \geq N,$$

means that

$$z_{1k} \in \Pi^{\overline{c}/2}(F, t_k), \quad k \geq N.$$

Since $\Pi^{\overline{c}/2}(F, t_k) \subset \operatorname{int} \Pi^{\overline{c}}(F, t_k)$, we obtain a contradiction with $z_{1k} \notin \operatorname{int} \Pi^{\overline{c}}(F, t_k)$. $\square$

3.5. Multivalued function $\mathbf{U}^c$.

For $c \geq 0$, $i = \overline{1, k}$, and $t \in T$, we set

$$\Pi^c_-(i, t) := \{x \in \mathbb{R}^n : x + \alpha B_i^{(3)}(t) \notin \Pi^c(i, t),\ \forall \alpha \geq 0\},$$

$$\Pi^c_+(i, t) := \{x \in \mathbb{R}^n : x + \alpha B_i^{(3)}(t) \notin \Pi^c(i, t),\ \forall \alpha \leq 0\}.$$

For each $i = \overline{1, k}$, on Z, let us introduce the scalar multivalued function

$$\mathbf{U}_i^c(t, x) := \begin{cases} \{-\mu_i\}, & x \in \Pi^c_-(i, t), \\ \{\mu_i\}, & x \in \Pi^c_+(i, t), \\ [-\mu_i, \mu_i], & x \in \Pi^c(i, t). \end{cases}$$

On Z, let us define the vector multivalued function

$$\mathbf{U}^c(t,x) := \begin{pmatrix} \mathbf{U}_1^c(t,x) \\ \mathbf{U}_2^c(t,x) \\ \vdots \\ \mathbf{U}_k^c(t,x) \end{pmatrix}.$$

4. Independence and Dependence Exponents of the Vectors $B_i^{(3)}(t)$

As is seen from Sec. 3, in considering a certain set of vectors $B_i^{(3)}(t)$, $i \in F$, on the interval T, the property of linear independence of the vectors $B_i^{(3)}(t)$, $i \in F$, for any $t \in T$ is useful. However, we have no assumptions that guarantee such a property. Moreover, such assumptions are not natural. For example, the linear independence is violated if at least one vector $B_i^{(3)}(t)$ of the set considered vanishes at a certain instant $t \in T$. To differentiate the cases close to the case of linear independence from the cases of rough linear independence, we need several concepts. We formulate them for an arbitrary set of vectors in $\mathbb{R}^n$ but will apply them to the vectors $B_i^{(3)}(t)$.

4.1. Independence and dependence exponents. In the space $\mathbb{R}^n$, let us consider a finite set $\mathcal{B}$ of vectors b_i, $i = \overline{1,s}$, $s \geq 1$. By the symbol $\mathcal{G}(\mathcal{B})$ we denote the subspace spanned by the set $\mathcal{B}$. Let $\alpha > 0$.

The set $\mathcal{B}$ for $s \geq 2$ is said to be linearly independent with exponent α if

$$d(b_i, \mathcal{G}(\mathcal{B} \setminus b_i)) \geq \alpha, \quad i = \overline{1,s}.$$

Note that the independence with exponent α implies the ordinary linear independence. Also, it is obvious that any finite set of linearly independent vectors is independent with a certain exponent $\alpha > 0$.

In the case where $s = 1$, by the linear independence with exponent α we mean the condition that the length of the vector considered is greater than or equal to α.

The set $\mathcal{B}$ for $s \geq 2$ is said to be dependent with exponent α if, among the vectors of a given set, there exists a vector b_{i_*} such that

$$d(b_{i_*}, \mathcal{G}(\mathcal{B} \setminus b_{i_*})) \leq \alpha,$$

i.e., there exists a vector α-close to the set of other vectors.

For $s = 1$, the dependence with exponent α means that the length of the vector considered does not exceed α.

Let us formulate the following assertion on the estimate for closeness of a given vector to the set of other vectors.

Lemma 4.1. *Let a set $\mathcal{B}$ of vectors $b_i \in \mathbb{R}^n$, $i = \overline{1,s}$, that are linearly independent with exponent α be given. Add one more vector $a \in \mathbb{R}^n$ to it. Assume that the extended set $\mathcal{B} \cup a$ is dependent with exponent $p < \alpha$. Then the vector a is ζ-close to the set $\mathcal{B}$ with the characteristic $\zeta = (1 + 2D/\alpha)p$, where D is the maximum of the lengths of $s+1$ vectors from the set $\mathcal{B} \cup a$.*

Proof. We denote by $\hat{\alpha}$ the maximum independence exponent of vectors of the set $\mathcal{B}$. Let $\hat{p}$ be the minimum dependence exponent of vectors of the set $\mathcal{B} \cup a$. The minimum characteristic of closeness of the vector a to the set $\mathcal{B}$ is denoted by $\hat{\zeta}$.

In the set $\mathcal{B} \cup a$, we distinguish a vector h that is $\hat{p}$-close to the set $(\mathcal{B} \cup a) \setminus h$ of other vectors.

1. If one can take the vector a as h, then we obtain

$$\hat{\zeta} = \hat{p} \leq p < \left(1 + \frac{2D}{\alpha}\right) p,$$

and the proof is completed.

2. Assume that the vector a cannot be taken as h.

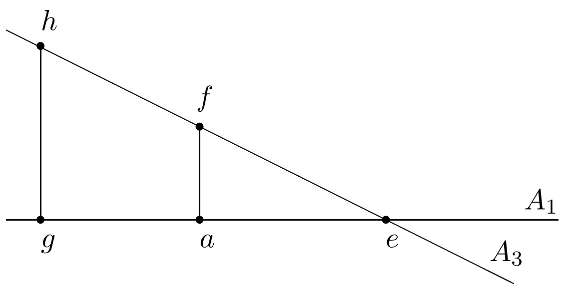

Fig. 1

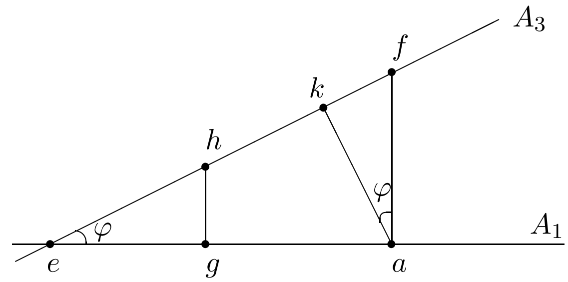

Fig. 2

A. First, consider the case where $s = 1$, i.e., the case where the set $\mathcal{B}$ consists of a single vector, which must be taken as the vector h. We have $\hat{\alpha} = |h|$. The relation $\hat{p}/\hat{\zeta} = |h|/|a|$ holds. Therefore,

$$\hat{\zeta} = \frac{|a|\hat{p}}{|h|} = \frac{|a|\hat{p}}{\hat{\alpha}} \leq \frac{|a|p}{\alpha} < \left(1 + \frac{2D}{\alpha}\right) p.$$

B. Now let $s > 1$. Consider the subspace $\mathcal{G}((\mathcal{B} \cup a) \setminus h)$. Let g be an element of this subspace nearest to h. We set $|hg| := |h - g|$. We have $|hg| = \hat{p} \leq p$. Also, we note that $d(h, \mathcal{G}(\mathcal{B} \setminus h)) \geq \alpha$.

Assume that $g \in \mathcal{G}(\mathcal{B} \setminus h)$. Then $|hg| \geq \alpha$. Then taking into account the inequality $p \geq |hg|$, we obtain $p \geq \alpha$, which contradicts the condition $p < \alpha$. Hence, $g \notin \mathcal{G}(\mathcal{B} \setminus h)$.

We have

$$a \notin \mathcal{G}(\mathcal{B} \setminus h), \quad a \in \mathcal{G}((\mathcal{B} \cup a) \setminus h),$$
$$g \in \mathcal{G}((\mathcal{B} \cup a) \setminus h), \quad g \notin \mathcal{G}(\mathcal{B} \setminus h) \subset \mathcal{G}((\mathcal{B} \cup a) \setminus h).$$

Consider the line A_1 passing through the points g and a. The line A_1 is either parallel to the subspace $\mathcal{G}(\mathcal{B} \setminus h)$ or intersects it.

If the line A_1 is parallel to the subspace $\mathcal{G}(\mathcal{B} \setminus h)$, then $d(a, \mathcal{G}(\mathcal{B})) \leq |hg|$. Indeed, drawing the line A_2 through the point h, which is parallel to A_1, we obtain that it is parallel to $\mathcal{G}(\mathcal{B} \setminus h)$. Hence $A_2 \subset \mathcal{G}(\mathcal{B})$. Therefore, $d(a, \mathcal{G}(\mathcal{B})) \leq d(a, A_2) \leq |hg|$.

Taking into account the relation $|hg| = \hat{p}$, from the inequality $d(a, \mathcal{G}(\mathcal{B})) \leq |hg|$, we obtain $d(a, \mathcal{G}(\mathcal{B})) \leq \hat{p}$, i.e., the vector a is $\hat{p}$-close to the set $\mathcal{B}$. This contradicts the assumption that the vector a cannot be taken as h.

Now let us assume that the line A_1 intersects the subspace $\mathcal{G}(\mathcal{B} \setminus h)$. Let e be the intersection point. Let us draw the line A_3 through the points h and e. We have $A_3 \subset \mathcal{G}(\mathcal{B})$. Let f be a point on the line A_3 such that fa is parallel to hg. Note that $|he| \geq \alpha$.

Let us consider the following variants of location of the point e on the line A_1.

(a) The point e lies on the ray ga with vertex g being more distant than the point a (Fig. 1). Then $d(a, \mathcal{G}(\mathcal{B})) \leq |fa| < |hg| = \hat{p}$, and this contradicts the assumption that the point a cannot be taken as h.

(b) The point e lies on the ray ag with vertex a being more distant than the point g (Fig. 2). We have

$$|he| \geq d(h, \mathcal{G}(\mathcal{B} \setminus h)) \geq \alpha.$$

Denote by k the point on the ray ef nearest to the point a. Note that $k \in \mathcal{G}(\mathcal{B})$. Hence $d(a, \mathcal{G}(\mathcal{B})) \leq |ka|$. Let us estimate $|ka|$.

We have (see Fig. 2) $|ka|/|fa| = \cos\varphi = |eg|/|eh|$. Therefore,

$$|ka| = |fa||eg|/|eh|. \tag{4.1}$$

Since $|fa|/|hg| = |ea|/|eg|$, it follows that

$$|fa| = \frac{|hg||ea|}{|eg|} = \frac{|hg|(|eg| + |ga|)}{|eg|} = |hg| \left(1 + \frac{|ga|}{|eg|}\right). \tag{4.2}$$

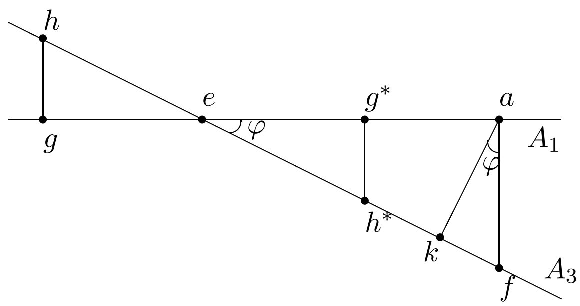

Fig. 3

Using (4.1) and (4.2), we obtain

$$|ka| = |fa|\frac{|eg|}{|eh|} = |hg|\left(1 + \frac{|ga|}{|eg|}\right)\frac{|eg|}{|eh|} = \frac{|hg|}{|eh|}\big(|eg| + |ga|\big).$$

Taking the inequalities $|eg| < |eh|$ and $|ga| < |ha|$ into account, we further have

$$|ka| < \frac{|hg|}{|eh|}\big(|eh| + |ha|\big).$$

Since $|hg| \leq p$, $|eh| \geq \alpha$, and $|ha| < 2D$, it follows that

$$|ka| < p\left(1 + \frac{|ha|}{\alpha}\right) < \left(1 + \frac{2D}{\alpha}\right)p.$$

Therefore, as the nearest characteristic of the vector a to the set $\mathcal{B}$, we can take $\zeta = |ka| < (1+2D/\alpha)p$.

(c) The point e lies on the ray ag between the points g and a (Fig. 3).

If $|ea| \leq |eg|$, then $|fa| \leq |hg|$, which contradicts the assumption that a cannot be taken as h.

Let $|ea| > |eg|$. Introduce the points g^* and h^* symmetric to the points g and h with respect to e. Let k be the point on the ray ef nearest to a. Let us estimate $|ka|$.

We proceed as in case (b), taking h^* and g^* instead of h and g. We have

$$|ka| = |fa|\frac{|eg^*|}{|eh^*|} = \frac{|h^*g^*|}{|eh^*|}(|eg^*| + |g^*a|).$$

Taking into account that $|hg| = |h^*g^*|$, $|eh| = |eh^*|$, and $|eg^*| + |g^*a| < |ga| < |ha|$, we then obtain

$$|ka| < \frac{|hg|}{|eh|}|ha|.$$

As a result,

$$|ka| < \frac{2Dp}{\alpha},$$

and, therefore, as the characteristic of closeness of the vector a to the set $\mathcal{B}$ in the case considered, we can take

$$\zeta = |ka| < \frac{2Dp}{\alpha} < \left(1 + \frac{2D}{\alpha}\right)p.$$

The lemma is proved. $\square$

4.2. Independence and dependence exponents of the vectors $B_i^{(3)}(t)$ on intervals of time. Each of the functions $B_i^{(3)}$, $i = \overline{1,k}$, satisfies the Lipshitz condition on T. The following property is a consequence of this property.

Let $\mathcal{T} \subset T$ be a closed interval and let $F \subset I$. Assume that for each $t \in \mathcal{T}$, the vectors $B_i^{(3)}(t)$, $i \in F$, are linearly independent. Let $\mathcal{G}^{\perp}(\{B_i^{(3)}(t)\}_{i\in F})$ be the orthogonal complement in $\mathbb{R}^n$ to the linear subspace $\mathcal{G}(\{B_i^{(3)}(t)\}_{i\in F})$ spanned by the vectors $B_i^{(3)}(t)$, $i \in F$. Then for any $j = \overline{1,k}$, the projection of the vector $B_j^{(3)}(t)$ on the subspace $\mathcal{G}^{\perp}(\{B_i^{(3)}(t)\}_{i\in F})$ satisfies the Lipschitz condition in t on the closed interval $\mathcal{T}$.

In turn, this property implies the existence of a lower estimate for the length of the interval on which the independence of the vectors $B_i^{(3)}(t)$, $i \in F$, with exponent α decreases up to the independence with exponent $\alpha/2$. Let us present the precise statement.

Proposition 4.1. *Let $F \subset I$. Then for any $\alpha > 0$, there exists $\kappa > 0$ such that if at a certain instant $t_* \in T$, the vectors $B_i^{(3)}(t)$ are independent with an exponent α, then they are independent with the exponent $\alpha/2$ for any $t \in [t_* - \kappa, t_* + \kappa] \cap T$.*

Analogously, we can formulate an assertion on the existence of a lower estimate of the length of the interval on which the dependence with an exponent α of the vectors $B_i^{(3)}(t)$, $i \in F$, can decrease up to the dependence with the exponent 2α.

Proposition 4.2. *Let $F \subset I$. Then for any $\alpha > 0$, there exists $\kappa > 0$ such that if at a certain instant $t_* \in T$, the vectors $B_i^{(3)}(t)$ are dependent with an exponent α, then they are dependent with the exponent 2α for any $t \in [t_* - \kappa, t_* + \kappa] \cap T$.*

4.3. Rule for choosing the independence and dependence exponents of the vectors $B_i^{(3)}(t)$. Let $\beta > 0$, $\xi > 0$, and $F \subset I$.

The independence (dependence) exponent of the vectors $B_i^{(3)}(t)$, $i \in F$, will be introduced only in the case where the number $q(F)$ of elements of the set F does not exceed n. We set the value of the exponent to be equal to $\xi^{q(F)}$.

At a certain instant of time $t_* \in T$, let the set of vectors $B_i^{(3)}(t_*)$, $i \in F$, be independent with exponent $\xi^{q(F)}$. Then, using Proposition 4.1, we can estimate from below the length of the closed interval $[t_*, \overline{t}] \subset T$, for each instant t such that the independence exponent of the set of vectors $B_i^{(3)}(t)$, $i \in F$, is not less than $\xi^{q(F)}/2$. Let $\overline{w}(F,\xi)$ be such an estimate. Since the total number of sets F with number of elements varying from 1 to $\min\{k,n\}$ is finite, we can choose the universal estimate

$$\overline{\overline{w}}(\xi) := \min_F \overline{w}(F,\xi).$$

We will apply it to any set F.

At a certain instant $t_* \in T$, let the set $B_i^{(3)}(t_*)$, $i \in F$, be dependent with exponent $\xi^{q(F)}$. Then using Proposition 4.2, we can estimate from below the length of the interval $[t_*, \hat{t}] \subset T$ for each instant t from which the dependence exponent of the vectors $B_i^{(3)}(t)$, $i \in F$, is not greater than $2\xi^{q(F)}$. Let $\widehat{w}(F,\xi)$ be such an estimate. By the finiteness of the number of sets F, we can choose the universal estimate

$$\widehat{\widehat{w}}(\xi) := \min_F \widehat{w}(F,\xi).$$

Denote

$$w(\xi) := \min\{\overline{\overline{w}}(\xi),\ \widehat{\widehat{w}}(\xi)\}.$$

The following property holds: for any F with number of elements $q(F)$ not exceeding $\min\{k,n\}$, on the interval of length $w(\xi)$, we guarantee the independence (dependence) of the vectors $B_i^{(3)}(t)$, $i \in F$, with the exponent $\xi^{q(F)}/2$ (resp. $2\xi^{q(F)}$) if it was with exponent $\xi^{q(F)}$ at the initial instant of time.

For $k > 1$, we only speak about the existence of the estimate $w(\xi)$. In the case $k = 1$, we can use the explicit estimate $w(\xi) = \xi/(2\beta)$.

4.4. Nearness estimate for the chosen exponents.

Lemma 4.2. *For a certain instant $t_* \in T$, let the set of vectors $B_i^{(3)}(t_*)$, $i \in F$, $q(F) < \min\{k, n\}$, be independent with exponent $\xi^{q(F)}$, where $\xi \leq 1/4$. Let us add one more vector $B_j^{(3)}(t_*)$, $j \notin F$. For the extended set, assume that there is the dependence with exponent $\xi^{q(F)+1}$. Then for any $t \in [t_*, t_*+w(\xi)] \cap T$, the vector $B_j^{(3)}(t)$ is close to the set $B_i^{(3)}(t_*)$, $i \in F$, with exponent $\zeta = ((1/2) + 8\sigma)\xi$.*

Proof. For the set of vectors $B_i^{(3)}(t)$, $i \in F$, with number of elements $q(F) < \min\{k, n\}$, on the closed interval $[t_*, t_*+w(\xi)] \cap T$, we have the independence with exponent $\xi^{q(F)}/2$. After adding the vector $B_j^{(3)}(t)$, $j \notin F$, the extended set on the closed interval $[t_*, t_* + w(\xi)] \cap T$ is dependent with exponent $2\xi^{q(F)+1}$.

Let us apply Lemma 4.1. We set

$$\alpha := \frac{\xi^{q(F)}}{2}, \quad p := 2\xi^{q(F)+1}.$$

The inequality $p < \alpha$ follows from $\xi \leq 1/4$. By Lemma 4.1, we obtain that for any $t \in [t_*, t_* + w(\xi)] \cap T$, the vector $B_j^{(3)}(t)$ is close to the set $B_i^{(3)}(t)$, $i \in F$, with the exponent

$$\zeta = \left(1 + \frac{2\sigma}{\alpha}\right) p = \left(1 + \frac{4\sigma}{\xi^{q(F)}}\right) 2\xi^{q(F)+1} = (2\xi^{q(F)} + 8\sigma)\xi < \left(2\left(\frac{1}{4}\right)^{q(F)} + 8\sigma\right)\xi \leq \left(\frac{1}{2} + 8\sigma\right)\xi.$$

In the latter inequality, the inequality $q(F) \geq 1$ is taken into account. ∎

5. Application of the Main Lemma

5.1. Weakening of inequality (2.3)**.** For a fixed ξ, we will use estimate (2.3) on closed intervals $[t_*, t^*]$ of length $t^* - t_* \leq w(\xi)$. The quantity δ in (2.3) is taken from the half-open interval $(0, t^* - t_*]$. The set of subscripts F is assumed to be chosen such that at each instant $t \in [t_*, t^*]$, the vectors $B_i^{(3)}(t)$, $i \in F$, are linearly independent with the exponent $\xi^{q(F)}/2$. We set

$$\mathcal{L} := (1/2) + 8\sigma, \quad \tilde{\zeta} := \mathcal{L}\xi.$$

Choose the set $H \subset I \setminus F$ such that $H \subset \mathcal{H}(F, \tilde{\zeta}, t)$ for all $t \in [t_*, t^*]$. Therefore, for any $j \in H$ and any $t \in [t_*, t^*]$, the vector $B_j^{(3)}(t)$ is $\tilde{\zeta}$-close to the set $B_i^{(3)}(t)$, $i \in F$. We assume that relations (2.2) hold for $t \in [t_*, t^*]$. As for the motion $y^{(1*)}(\cdot)$ of system (1.1) with certain admissible open-loop controls $u(\cdot)$ and $v(\cdot)$, it is assumed that at instant t_*, it emanates from the point x_*, and for any $i \in I \setminus (F \cup H)$, $t \in [t_* + \omega, t^*]$, in the case $x_* \in \Pi_+(i, t_*)$, the relation $u_i(t) = \mu_i$ holds, whereas in the case $x_* \in \Pi_-(i, t_*)$, the relation $u_i(t) = -\mu_i$ holds.

Let us rewrite estimate (2.3). We have

$$\begin{aligned} \mathcal{V}(F, t_* + \delta, y^{(1*)}(t_* + \delta)) &\leq V^{(2)}(t_*, x_*) + \lambda\delta^2 \sum_{i=1}^{k} \beta_i\mu_i \\ &+ 2\lambda\mathcal{L}\xi\delta \sum_{i\in H} \mu_i + 2\lambda\omega \sum_{i\notin F\cup H} \sigma_i\mu_i + \lambda\chi(t_*, t_* + \delta). \end{aligned} \tag{5.1}$$

In estimate (5.1), we weaken the third and fourth summands on the right-hand side:

$$2\lambda\mathcal{L}\xi\delta \sum_{i\in H} \mu_i \leq 2\lambda\mathcal{L}\mu\xi\delta, \quad 2\lambda\omega \sum_{i\notin F\cup H} \sigma_i\mu_i \leq 2\lambda\sigma\mu\omega.$$

We obtain the following inequality for any $\delta \in (0, t^* - t_*]$:

$$\mathcal{V}(F, t_* + \delta, y^{(1*)}(t_* + \delta)) \leq V^{(2)}(t_*, x_*) + \lambda\delta^2 \sum_{i=1}^{k} \beta_i \mu_i \\ +2\lambda\mathcal{L}\mu\xi\delta + 2\lambda\sigma\mu\omega + \lambda\chi(t_*, t_* + \delta). \tag{5.2}$$

The right-hand side of inequality (5.2) is independent of F. This inequality can be applied in the case where on the closed interval $[t_*, t^*]$, for $i \in F$, we use arbitrary admissible controls $u_i(\cdot)$, and for any $j \notin F$, either on $[t_* + \omega, t^*]$, the regular control $u_j(\cdot)$ acts, or for any $t \in [t_*, t^*]$, the vector $B_j^{(3)}(t)$ is $\tilde{\zeta}$-close to the set $B_i^{(3)}(t)$, $i \in F$. Recall that a control $u_j(\cdot)$ is said to be "regular" if for $x_* \in \Pi_+(j, t_*)$ ($x_* \in \Pi_-(j, t_*)$), on $[t_* + \omega, t^*]$, the relation $u_j(t) = \mu_j$ ($u_j(t) = -\mu_j$) holds.

We restrict the application of inequality (5.2) only to the case where $t^* - t_* \leq \mathcal{L}\xi/\beta$. Then replacing one factor δ in the second summand on the right-hand side by $\mathcal{L}\xi/\beta$, instead of (5.2) we obtain the estimate

$$\mathcal{V}(F, t_* + \delta, y^{(1*)}(t_* + \delta)) \leq V^{(2)}(t_*, x_*) + 3\lambda\mathcal{L}\mu\xi\delta + 2\lambda\sigma\mu\omega + \lambda\chi(t_*, t_* + \delta)$$

or, which is the same,

$$\mathcal{V}(F, t, y^{(1*)}(t)) \leq V^{(2)}(t_*, x_*) + \mathbf{kf}(\xi)(t - t_*) + 2\lambda\sigma\mu\omega + \lambda\chi(t_*, t), \quad t \in [t_*, t^*], \tag{5.3}$$

where

$$\mathbf{kf}(\xi) := 3\lambda\mathcal{L}\mu\xi.$$

5.2. Weakening of inequality (2.11)**.** We will apply inequality (2.11), setting $\zeta = 2\xi$. In this case, in (2.11), we weaken the second and third summands on the right-hand side:

$$4\lambda\xi \cdot (t - t_*) \sum_{i \in H} \mu_i \leq 4\lambda\mu\xi \cdot (t - t_*), \quad 2\lambda\omega \sum_{i \notin H} \sigma_i \mu_i \leq 2\lambda\sigma\mu\omega.$$

Keeping the substitution part of Proposition 2.1 the same, for $\zeta = 2\xi$, we obtain the estimate

$$V^{(2)}(t, y^{(1*)}(t)) \leq V^{(2)}(t_*, x_*) + \mathbf{kf}^{[*]}(\xi)(t - t_*) + 2\lambda\sigma\mu\omega + \lambda\chi(t_*, t), \quad t \in [t_*, t^*], \tag{5.4}$$

where

$$\mathbf{kf}^{[*]}(\xi) := 4\lambda\mu\xi.$$

6. Choice of c_h, Δ_h, and $\mathbf{st}^{[h]}$

We set

$$\mathrm{st}(\xi, \Delta, c) := \frac{2\lambda\sigma\mu\Delta + c}{\mathbf{kf}(\xi)}, \quad \Delta > 0, \quad c \geq 0, \quad \xi > 0.$$

In what follows, we will agree that

$$\xi \leq \frac{\sigma}{1 + 16\sigma}.$$

The following relations hold for such ξ:

$$\xi < \sigma, \quad \xi < 1/4, \quad \mathrm{st}(\xi, \Delta, c) > \Delta.$$

Consider the set of integers $h \geq 1$, $h \leq \min\{k, n\}$, for each of which there exist a set $F \subset I$ with number $q(F) = h$ of elements and an instant $t \in T$ such that the set of vectors $B_i^{(3)}(t)$, $i \in F$, is independent with the exponent ξ^h. Such a set is nonempty because of the inequality $\xi < \sigma$. Denote by $h^*(\xi)$ the maximum among these numbers.

We assume that the value of ξ is fixed.

1. Choose $c_{h^*(\xi)} > 0$ and $\Delta_{h^*(\xi)} > 0$ such that

$$\mathrm{st}(\xi, \Delta_{h^*(\xi)}, c_{h^*(\xi)}) \leq \min\left\{\frac{\mathcal{L}\xi}{\beta},\ w(\xi),\ \vartheta - \vartheta_1\right\}.$$

Below, for brevity, we agree to omit the argument ξ when it does not lead to confusion.

2. Let us pass to the choice of c_h and Δ_h for $h = h^* - 1, h^* - 2, \ldots, 1$.

Let us describe the step of induction. Let $c_h > 0$ and $\Delta_h > 0$ be introduced for a certain $h \in \overline{2, h^*}$. Let us define c_{h-1} and Δ_{h-1}.

A. Fix a set F_h of h elements. Choose an arbitrary subset F_{h-1} of $h-1$ elements and the remaining set F_1 of a single element. Let $\mathcal{K}(F_h, F_{h-1})$ be the set of instants of time $t \in T$ for each of which there is the ξ^h-independence of the vectors $B_i^{(3)}(t)$, $i \in F_h$ and, simultaneously, the ξ^{h-1}-independence of the vectors $B_i^3(t)$, $i \in F_{h-1}$.

In the case $\mathcal{K}(F_h, F_{h-1}) \neq \varnothing$, to each instant from the set $\mathcal{K}(F_h, F_{h-1})$, we put in correspondence the closed interval of length $w(\xi)$ adjacent to it to the right. If the right endpoint of such an interval is greater than ϑ, we take a closed interval with ϑ as the right endpoint. The union over $t \in \mathcal{K}(F_h, F_{h-1})$ of these intervals is denoted by $\widehat{\mathcal{K}}(F_h, F_{h-1})$. Note that $\widehat{\mathcal{K}}(F_h, F_{h-1})$ is a closed bounded set. At any instant $t \in \widehat{\mathcal{K}}(F_h, F_{h-1})$, we have the $\xi^h/2$-independence of the vectors $B_i^{(3)}(t)$, $i \in F_h$, and the $\xi^{h-1}/2$-independence of the vectors $B_i^{(3)}(t)$, $i \in F_{h-1}$. Also, we have the $\xi^h/2$-independence of the vectors $B_i^{(3)}(t)$, $i \in F_1$.

In the case $\mathcal{K}(F_h, F_{h-1}) = \varnothing$, we do not take into account the pair F_h, F_{h-1} for the definition of c_{h-1} and Δ_{h-1}. Below, we will assume that $\mathcal{K}(F_h, F_{h-1}) \neq \varnothing$.

Using Lemma 3.3, we choose $c_\sharp \in (0, c_h]$ so that, uniformly in $t \in \widehat{\mathcal{K}}(F_h, F_{h-1})$, the sets $\Pi^{c_\sharp}(F_{h-1}, t)$ and $\Pi^{c_\sharp}(F_1, t)$ outside the set $\operatorname{int} \Pi^{c_h}(F_h, t)$ are distant from each other by a finite distance in the limits of a certain bounded set $\mathcal{X}_* \subset \mathbb{R}^n$ estimated from above by the set of states at which motions of system (1.1) emanating from the set $\mathcal{Y}$ can lie.

Let $b(\xi, F_h, F_{h-1}, c_h, c)$, where $c \in (0, c_\sharp]$, be a uniform (in $\bar{t} \in \widehat{\mathcal{K}}(F_h, F_{h-1})$) lower estimate of the transition time of system (1.1) from the set $\Pi^c(F_{h-1}, \bar{t}) \cap \mathcal{X}_*$ to the set $\Pi^c(F_1, t)$, $t \in (\bar{t}, \bar{t} + w(\xi)] \cap T$, if at the initial instant $\bar{t}$, the system was on $(\Pi^c(F_{h-1}, \bar{t}) \cap \mathcal{X}_*) \setminus \operatorname{int} \Pi^{c_h}(F_h, \bar{t})$. Such an estimate exists by the property that the set $\Pi^c(F_1, t)$ is continuously varied in t, which was proved in Lemma 3.2. Obviously, the dependence $c \mapsto b(\xi, F_h, F_{h-1}, c_h, c)$ can be chosen to be nonincreasing. For example, we can take $b(\xi, F_h, F_{h-1}, c_h, c) = b(\xi, F_h, F_{h-1}, c_h, c_\sharp)$, $c \in (0, c_\sharp]$.

Choose positive $c_\flat \leq c_\sharp$ and $\Delta_\flat \leq \Delta_h$ such that

$$\operatorname{st}(\xi, \Delta_\flat, c_\flat) \leq b(\xi, F_h, F_{h-1}, c_h, c_\flat).$$

As a result, we obtain $c_\flat$ and $\Delta_\flat$ for given F_h and F_{h-1}. To stress the dependence of the chosen quantities on ξ, F_h, F_{h-1}, c_h, and Δ_h, we will write $c_\flat(\xi, F_h, F_{h-1}, c_h, \Delta_h)$ and $\Delta_\flat(\xi, F_h, F_{h-1}, c_h, \Delta_h)$.

B. Now we look over all sets F_h of h elements, and for each of them we look over all the sets F_{h-1} of $h-1$ elements (with the condition $\mathcal{K}(F_h, F_{h-1}) \neq \varnothing$ holding). We set

$$c_{h-1} := \min_{(F_h, F_{h-1})} c_\flat(\xi, F_h, F_{h-1}, c_h, \Delta_h), \quad \Delta_{h-1} := \min_{(F_h, F_{h-1})} \Delta_\flat(\xi, F_h, F_{h-1}, c_h, \Delta_h).$$

If, for all variants of F_h and F_{h-1}, we have $\mathcal{K}(F_h, F_{h-1}) = \varnothing$, then we set $c_{h-1} = c_h$ and $\Delta_{h-1} = \Delta_h$.
Denote

$$\mathbf{st}^{[h]} := \operatorname{st}(\xi, \Delta_h, c_h), \quad h = \overline{1, h^*}.$$

The quantity $\mathbf{st}^{[h]}$ depends on ξ. For brevity, we omit the bracket of the argument.

7. Loops along the Motion

Therefore, for a given ξ, we have introduced the quantities c_h, Δ_h, and $\mathbf{st}^{[h]}$, $h = \overline{1, h^*}$.

Consider the motion $y^{(1)}(\cdot)$ of system (1.1) from the position $(t_0, x_0) \in \mathcal{Y}$, $t_0 < \vartheta$, according to a certain strategy $U \in \mathbf{U}^{c_1}$ of the first player with a step $\Delta \leq \Delta_1$ and according to a certain open-loop control $v(\cdot)$ of the second player.

By the symbol $\mathfrak{H}$ we denote the set of integers $h \in \overline{1, h^*}$, for each of which there exist a set $F \subset I$ with number of elements $q(F) = h$ and an instant $t \in [t_0, \vartheta]$ such that $y^{(1)}(t) \in \Pi^{c_h}(F, t)$ and the set of vectors $B_i^{(3)}(t)$, $i \in F$, is independent with exponent ξ^h.

If $\mathfrak{H} = \varnothing$, then we agree that the whole interval $[t_0, \vartheta]$ is a free interval of level 1.

Assume that $\mathfrak{H} \neq \varnothing$. Let $h^\flat$ be the maximum among the set of numbers in $\mathfrak{H}$.

Along the motion $y^{(1)}(\cdot)$, we distinguish "loops," which are connected to the entrance to the sets $\Pi^{c_h}(F, t)$, where $q(F) = h$, $h = \overline{1, h^\flat}$. Also, we define free intervals.

Let $\mathfrak{T}^{[h]}$ be the set of instants $t \in [t_0, \vartheta]$ such that $y^{(1)}(t) \in \Pi^{c_h}(F, t)$ for a certain set F with $q(F) = h$, and, moreover, the vectors $B_i^{(3)}(t)$, $i \in F$, are independent with exponent ξ^h. The mentioned F is not necessarily unique. The collection of sets F corresponding to the instant t is denoted by $\{F^{[h]}(t)\}$.

If $h^\flat < h^*$, then we assume that $[t_0, \vartheta]$ is a free interval of level $h^\flat + 1$.

1. Let $h = h^\flat$.

A. Let us agree that

$$\tau_1^{[h^\flat]} := \min\{t : t \in \mathfrak{T}^{[h^\flat]}\},$$
$$\tau_{i+1}^{[h^\flat]} := \min\Big\{t : t \in \mathfrak{T}^{[h^\flat]} \cap [\tau_i^{[h^\flat]} + \mathbf{st}^{[h^\flat]}, \vartheta]\Big\}, \quad i = 1, 2, \ldots.$$

The obtained set of instants $\tau_i^{[h^\flat]}$ is denoted by $\{\tau^{[h^\flat]}\}$.

To each instant $\tau_i^{[h^\flat]} \in \{\tau^{[h^\flat]}\}$, we put in correspondence a series of loops. We set $t_{i,1}^{[h^\flat]} := \tau_i^{[h^\flat]}$. The instant $t_{i,1}^{[h^\flat]}$ is called the instant of the origin for the first loop (in the series i) of level $h^\flat$. Choose an arbitrary $F \in \{F^{[h^\flat]}(t_{i,1}^{[h^\flat]})\}$. We define the instant $t_{i,1+}^{[h^\flat]}$ of the end of the first loop as follows:

$$t_{i,1+}^{[h^\flat]} := \max\Big\{t : y^{(1)}(t) \in \Pi^{c_{h^\flat}}(F, t),\ t \in [t_{i,1}^{[h^\flat]}, \tau_i^{[h^\flat]} + \mathbf{st}^{[h^\flat]}] \cap [t_0, \vartheta]\Big\}.$$

We do not verify the independence property of the vectors $B_i^{(3)}(t)$, $i \in F$, with exponent $\xi^{h^\flat}$. In particular, the instant $t_{i,1+}^{[h^\flat]}$ can coincide with $t_{i,1}^{[h^\flat]}$.

As the instant $t_{i,2}^{[h^\flat]}$ of the origin of the second loop (in the series i) of level $h^\flat$, we take

$$t_{i,2}^{[h^\flat]} := \min\Big\{t : t \in \mathfrak{T}^{[h^\flat]} \cap [t_{i,1+}^{[h^\flat]},\ \tau_i^{[h^\flat]} + \mathbf{st}^{[h^\flat]}]\Big\}.$$

Choose an arbitrary $F \in \{F^{[h^\flat]}(t_{i,2}^{[h^\flat]})\}$. Define the instant $t_{i,2+}^{[h^\flat]}$ of the end of the second loop:

$$t_{i,2+}^{[h^\flat]} := \max\Big\{t : y^{(1)}(t) \in \Pi^{c_{h^\flat}}(F, t),\ t \in [t_{i,2}^{[h^\flat]},\ \tau_i^{[h^\flat]} + \mathbf{st}^{[h^\flat]}] \cap [t_0,\ \vartheta]\Big\}.$$

Continuing this process, we obtain the series of loops of level $h^\flat$ corresponding to the instant $\tau_i^{[h^\flat]}$. The number of loops in the series does not exceed $C_k^{h^\flat}$.

Enumerate the loops of level $h^\flat$ in through numbering. The set of instants of origins of loops of level $h^\flat$ is denoted by $\{t^{[h^\flat]}\}$.

B. Remove the intervals of the constructed loops from the closed interval $[t_0, \vartheta]$. We obtain an ordered set of intervals. Each of them can be closed and is called a free interval of level $h^\flat$. The origin of the first free interval can coincide with t_0, and the endpoint of the last free interval can coincide with ϑ. Let $\Theta_j^{[h^\flat]}$ be the notation of the free interval with the number j in the through numbering.

2. Now we pass to levels $h^\flat - 1, h^\flat - 2, \ldots, 1$. Let us describe a step of induction.

Assume that the loops and free intervals corresponding to a certain level $h \in \overline{2, h^\flat}$ are already defined. Let $\Theta_j^{[h]} := [\theta_j^{[h]}, \theta_{j+}^{[h]}]$ be the notation of the free interval with the number j in the through numbering on $[t_0, \vartheta]$.

A. Consider a free interval $\Theta_j^{[h]}$. Let us agree that

$$\tau_{j,1}^{[h-1]} := \min\big\{t : t \in \mathfrak{T}^{[h-1]} \cap \Theta_j^{[h]}\big\},$$
$$\tau_{j,i+1}^{[h-1]} := \min\Big\{t : t \in \mathfrak{T}^{[h-1]} \cap [\tau_{j,i}^{[h-1]} + \mathbf{st}^{[h-1]},\ \vartheta_{j+}^{[h]}]\Big\}, \quad i = 1, 2, \ldots.$$

The obtained set of instants $\tau_{j,i}^{[h-1]}$ is denoted by $\{\tau_j^{[h-1]}\}$.

To each instant $\tau_{j,i}^{[h-1]} \in \{\tau_j^{[h-1]}\}$, we put in correspondence a series of loops. We set $t_{j,i,1}^{[h-1]} := \tau_{j,i}^{[h-1]}$. An instant $t_{j,i,1}^{[h-1]}$ is called the instant of origin of the first loop (in the series i) of level $h-1$ on the free interval $\Theta_j^{[h]}$. Choose an arbitrary $F \in \{F^{[h-1]}(t_{j,i,1}^{[h-1]})\}$. Define the instant $t_{j,i,1+}^{[h-1]}$ of the end of the first loop as follows:

$$t_{j,i,1+}^{[h-1]} := \max\Big\{t : y^{(1)}(t) \in \Pi^{c_{h-1}}(F, t),\ t \in [t_{j,i,1}^{[h-1]},\ \tau_{j,i}^{[h-1]} + \mathbf{st}^{[h-1]}] \cap \Theta_j^{[h]}\Big\}.$$

In particular, the instant $t_{j,i,1+}^{[h-1]}$ can coincide with $t_{j,i,1}^{[h-1]}$. Further, we find the instant $t_{j,i,2}^{[h-1]} \in [t_{j,i,1+}^{[h-1]},\ \tau_{j,i}^{[h-1]} + \mathbf{st}^{[h-1]}] \cap \Theta_j^{[h]}$, and so on.

B. Define free intervals of level $h-1$ on $\Theta_j^{[h]}$. For this purpose, we remove the intervals of the constructed loops of level $h-1$ constructed on the closed interval $\Theta_j^{[h]}$ from it. Each of the remaining intervals can be closed and is called a free interval. The origin of the first free interval can coincide with $\theta_j^{[h]}$, and the endpoint of the last free interval can coincide with $\theta_{j+}^{[h]}$.

C. Using the method described, we introduce loops of level $h-1$ and free intervals of level $h-1$ on each free interval $\Theta_j^{[h]}$ of level h.

Perform through enumeration of instants $\tau_{j,i}^{[h-1]}$, running through all values of the subscripts j and i. The obtained set of instants is denoted by $\{\tau^{[h-1]}\}$.

Also, in the through numbering, we enumerate the instants of origin of the loops of level $h-1$. Denote such a set by $\{t^{[h-1]}\}$. Introduce the through numbering of free intervals. Let $\Theta_j^{[h-1]} := [\theta_j^{[h-1]}, \theta_{j+}^{[h-1]}]$ be the notation of the free interval with the number j.

Note that the free intervals on the level $h-1$ can be defined as a result of the removal of all the loops of levels $h^\flat$, $h^\flat - 1, \ldots, h-1$ from $[t_0, \vartheta]$ and the subsequent closing each of the obtained intervals.

If there are no loops on the level $h-1$, then going around the level $h-1$, we pass to the formation of loops on the level $h-2$. In this case, we assume that free intervals of level $h-1$ coincide with free intervals of level h.

8. Proof of Theorem 1 for $\beta > 0$

In Sec. 7, for a fixed ξ, to a motion $y^{(1)}(\cdot)$, we have put in correspondence loops and free intervals of levels $1, 2, \ldots, h^\flat$. The set of instants of origin of the loops of level h was denoted by $\{t^{[h]}\}$.

To write the variation of the function $V^{(2)}$ along the motion $y^{(1)}(\cdot)$ on the interval $[t_*, t^*]$, we introduce the notation

$$\mathrm{Var}(V^{(2)}, [t_*, t^*]) := V^{(2)}(t^*, y^{(1)}(t^*)) - V^{(2)}(t_*, y^{(1)}(t_*)).$$

We set

$$\mathbf{sh}^{[*]} := 2\lambda\sigma\mu\Delta; \quad \mathbf{sh}^{[h]} := 2\lambda\sigma\mu\Delta + c_h, \quad h = \overline{1, h^\flat}.$$

The quantity $\mathbf{sh}^{[*]}$ depends on Δ, and the quantity $\mathbf{sh}^{[h]}$ depends on ξ and Δ, but we omit the brackets of arguments.

From the definition of $\mathbf{st}^{[h]}$ and $\mathbf{sh}^{[h]}$ and the condition $\Delta \le \Delta_h$, the following relation follows:

$$\mathbf{sh}^{[h]} \le \mathbf{st}^{[h]} \cdot \mathbf{kf}, \quad h = \overline{1, h^\flat}.$$

Further, let

$$\mathbf{kf}^{[0]} := \max\{\mathbf{kf}, \mathbf{kf}^{[*]}\}; \quad \mathbf{kf}^{[h]} := e(h,k)\mathbf{kf}^{[0]}, \quad h = \overline{1, h^\flat},$$

where

$$e(1,k) := 2C_k^1 + 1, \quad e(h,k) := e(h-1,k) + C_k^h(1 + e(h-1,k)), \quad h = \overline{2, h^\flat}.$$

Note that

$$\mathbf{sh}^{[*]} < \mathbf{sh}^{[1]}; \quad \mathbf{sh}^{[h-1]} \leq \mathbf{sh}^{[h]}, \quad h = \overline{2, h^\flat},$$
$$\mathbf{kf}^{[0]} < \mathbf{kf}^{[1]}; \quad \mathbf{kf}^{[h-1]} < \mathbf{kf}^{[h]}, \quad h = \overline{2, h^\flat}.$$

8.1. Increment of the function $V^{(2)}$ on loops of level h. Let us explain the application of inequality (5.3), which follows from the main Lemma 2.1, in order to estimate the variation of the function $V^{(2)}$ along the motion $y^{(1)}(\cdot)$ on loops of level h.

Any instant $t_j^{[h]}$ from $\{t^{[h]}\}$ is the instant of origin of a loop of the hth level, the instant of the end of a loop is denoted by $t_{j+}^{[h]}$, and, moreover, $t_{j+}^{[h]} - t_j^{[h]} \leq \mathbf{st}^{[h]}$. To each loop of level h, we put in correspondence a completely definite set F of subscripts with number of elements $q(F) = h$. At the instant $t_j^{[h]}$, we have the ξ^h-independence of the vectors $B_i^{(3)}(t_j^{[h]})$, $i \in F$. On the interval $[t_j^{[h]}, t_{j+}^{[h]}]$, the independence with exponent ξ^h can decrease only to the independence with exponent $\xi^h/2$.

To apply inequality (5.3) for estimating the value of $\mathcal{V}(F, t_{j+}^{[h]}, y^{(1)}(t_{j+}^{[h]}))$, using Lemma 4.2 we define the set $H(F, \mathcal{L}\xi)$ consisting of subscripts $g \notin F$ of the vectors $B_g^{(3)}(t)$ that are close with the characteristic $\mathcal{L}\xi$, $\mathcal{L} = (1/2) + 8\sigma$, to the set of vectors $B_i^{(3)}(t)$, $i \in F$, for all t on the interval $[t_j^{[h]}, t_{j+}^{[h]}]$. Taking an element $g \notin F$, we include it in the set $H(F, \mathcal{L}\xi)$ if, at the instant of time $t_j^{[h]}$, there is the ξ^{h+1}-dependence of the vectors $B_i^{(3)}(t_j^{[h]})$, $i \in F \cup g$.

Let $g \notin F$, $g \notin H(F, \mathcal{L}\xi)$. For these elements, we have the ξ^{h+1}-independence of the vectors $B_i^{(3)}(t_j^{[h]})$, $i \in F \cup g$. In this case, we take into account that, by the definitions of the sets $\{t^{[h+1]}\}$ and $\{t^{[h]}\}$, the instant $t_j^{[h]}$ belongs to a certain free interval of level $h+1$. In the interior of this interval, there are no loops of level $h+1$. Therefore,

$$y^{(1)}(t_j^{[h]}) \notin \operatorname{int} \Pi^{c_{h+1}}(F \cup g, t_j^{[h]}).$$

Since $t_{j+}^{[h]} - t_j^{[h]} \leq \mathbf{st}^{[h]}$, it follows that on $[t_j^{[h]}, t_{j+}^{[h]}]$, we have the independence of the vectors $B_i^{(3)}(t)$, $i \in F \cup g$, with exponent $\xi^{h+1}/2$. Hence, using the rule for choosing the numbers c_{h+1}, c_h, and c_1 described in Sec. 6 and based on Lemma 3.3, we can speak about a uniform lower estimate of the distance between the sets $(\Pi^{c_h}(F,t) \cap \mathcal{X}_*) \setminus \operatorname{int} \Pi^{c_{h+1}}(F \cup g, t)$ and $\Pi^{c_1}(g,t)$ for instants $t \in T$, at which the vectors $B_i^{(3)}(t)$, $i \in F \cup g$, are independent with exponent $\xi^{h+1}/2$. Therefore, the distance between the point $y^{(1)}(t_j^{[h]})$ and the set $\Pi^{c_1}(g, t_j^{[h]})$ also satisfies this estimate. According to the choice of $\mathbf{st}^{[h]}$, the motion of system (1.1) emanating from the point $y^{(1)}(t_j^{[h]})$ at the instant $t_j^{[h]}$ cannot attain the set $\Pi^{c_1}(g,t)$, continuously varying on the interval $[t_j^{[h]}, t_{j+}^{[h]}]$ by Lemma 3.2.

Moreover, on $[t_j^{[h]}, t_{j+}^{[h]}]$, along the motion $y^{(1)}(\cdot)$, the regular control $u_g(\cdot)$ is realized everywhere, probably, except for a certain initial interval adjacent to the instant $t_j^{[h]}$ whose length does not exceed the step $\Delta \leq \Delta_1$ of the discrete scheme.

Therefore, on the interval $[t_j^{[h]}, t_{j+}^{[h]}]$, we can apply the main Lemma 2.1 and the inequalities that follow from it. In this case, we set $\omega = \Delta$.

Using inequality (5.3), we write the upper estimate of $\mathcal{V}(F, t_{j+}^{[h]}, y^{(1)}(t_{j+}^{[h]}))$:

$$\mathcal{V}(F, t_{j+}^{[h]}, y^{(1)}(t_{j+}^{[h]})) \leq V^{(2)}(t_j^{[h]}, y^{(1)}(t_j^{[h]})) + \mathbf{kf} \cdot (t_{j+}^{[h]} - t_j^{[h]}) + 2\lambda\sigma\mu\Delta + \lambda\chi(t_j^{[h]}, t_{j+}^{[h]}).$$

Since $y^{(1)}(t^{[h]}_{j+}) \in \Pi^{c_h}(F, t^{[h]}_{j+})$, we have the following inequality for the function $V^{(2)}$:

$$V^{(2)}(t^{[h]}_{j+}, y^{(1)}(t^{[h]}_{j+})) \leq \mathcal{V}(F, t^{[h]}_{j+}, y^{(1)}(t^{[h]}_{j+})) + c_h.$$

Therefore,

$$\mathrm{Var}(V^{(2)}, [t^{[h]}_j, t^{[h]}_{j+}]) \leq \mathbf{kf} \cdot (t^{[h]}_{j+} - t^{[h]}_j) + 2\lambda\sigma\mu\Delta + c_h + \lambda\chi(t^{[h]}_j, t^{[h]}_{j+}).$$

Taking into account that $\mathbf{sh}^{[h]} = 2\lambda\sigma\mu\Delta + c_h$, we obtain

$$\mathrm{Var}(V^{(2)}, [t^{[h]}_j, t^{[h]}_{j+}]) \leq \mathbf{kf} \cdot (t^{[h]}_{j+} - t^{[h]}_j) + \mathbf{sh}^{[h]} + \lambda\chi(t^{[h]}_j, t^{[h]}_{j+}). \tag{8.1}$$

8.2. Auxiliary intervals. Let us define variants of intervals of time using which we will estimate the variation of the function $V^{(2)}$ along the motion $y^{(1)}(\cdot)$.

Let us agree that each of the subscripts h and p assumes the values $1, 2, \dots, h^\flat$. Let

- $\mathbf{E}^{[h]}$ be the interval $[\rho, \eta]$ such that it contains at least one instant from the set $\{\tau^{[h]}\}$; on $[\rho, \eta)$, there are no points of the sets $\{t^{[p]}\}$, $p > h$;
- $E^{[h]}$ is the interval $[\rho, \eta]$ such that $\rho \in \{\tau^{[h]}\}$; on $[\rho, \eta] \setminus [\rho, \rho + \mathbf{st}^{[h]}]$, there are no points of the set $\{\tau^{[h]}\}$; on $[\rho, \eta)$, there are no points of the sets $\{t^{[p]}\}$, $p > h$;
- $\mathcal{E}^{[h]}$ is the interval $[\rho, \eta]$ of the form $E^{[h]}$ satisfying the additional condition $\rho + \mathbf{st}^{[h]} \leq \eta$.

We denote by $\mathcal{D}$ the interval $[\rho, \eta]$ if it is one of the free intervals of level 1. On (ρ, η), there are no instants from $\{t^{[p]}\}$, $p \geq 1$. This follows from the fact that free intervals of level 1 are obtained by removing all loops of levels $h^\flat, h^\flat - 1, \dots, 1$, from the closed interval $[t_{,0}, \vartheta]$ and by subsequently closing each of the remaining intervals.

On each interval of the form $E^{[h]}$ or $\mathcal{E}^{[h]}$, there is at least one instant of the set $\{t^{[h]}\}$. Also, there can be instants of the sets $\{t^{[p]}\}$, $p < h$.

All initial instants of h-loops, $h = \overline{1, h^\flat}$, that lie in a concrete interval $E^{[h]}$ are on an interval of length no more than $\mathbf{st}^{[h]}$. Moreover, on such an interval, for each F with $q(F) = h$, there can be no more than one instant of the loop origin. Therefore, the total number of h-loops on $E^{[h]}$ does not exceed C^h_k. Also, note that the intervals of h-loops lying in $E^{[h]}$ are disjoint. Hence we can perform an independent summation of estimate (8.1) over h-loop intervals lying in $E^{[h]}$, and we know the estimate of the number of loops.

8.3. Increment of the function $V^{(2)}$ on auxiliary intervals. Let us estimate the variation of the function $V^{(2)}$ on auxiliary intervals.

We begin with the interval $[\rho, \eta]$ of the form $\mathcal{D}$. We note the following properties:

- if on (ρ, η), there are entrances to the sets $\Pi^{c_1}(i, t)$, $i = \overline{1, k}$, then at each such instant, we have the inequality $|B^{(3)}_i(t)| < \xi$;
- a possible increase in the function $V^{(2)}$ due to "irregular" controls at one step of the discrete scheme is estimated from above by $\mathbf{sh}^{[*]}$ by virtue of Proposition 2.2.

Lemma 8.1. *The increment of the function $V^{(2)}$ on the interval $[\rho, \eta]$ of the form $\mathcal{D}$ is described by the inequality*

$$\mathrm{Var}_{\mathcal{D}}(V^{(2)}, [\rho, \eta]) \leq \mathbf{kf}^{[*]} \cdot (\eta - \rho) + \mathbf{sh}^{[*]} + \lambda\chi(\rho, \eta). \tag{8.2}$$

Proof. Note that

$$\Delta \leq \Delta_1 \leq w(\xi) \leq \xi/(2\beta).$$

Choose $\overline{\delta} := \xi/(2\beta)$. Then for time $\overline{\delta}$, the quantity $|B^{(3)}_i(t)|$, $i = \overline{1, k}$, cannot change more than $\xi/2$. Moving from left to right, we divide the interval $[\rho, \eta]$ with step $\overline{\delta}$ (the last of the obtained intervals can be of length less than $\overline{\delta}$). Let us show that we can apply Proposition 2.1 with estimate (5.4) to each of the $\overline{\delta}$-intervals.

Indeed, let $[\overline{t}, \tilde{t}]$ be an arbitrary interval from the $\overline{\delta}$-intervals. By the choice of the number $\overline{\delta}$, we obtain that for any $i \in \overline{1,k}$, either $|B_i^{(3)}(t)| \leq 2\xi$ for all $t \in [\overline{t}, \tilde{t}]$ or $|B_i^{(3)}(t)| \geq 3\xi/2$ also for all $t \in [\overline{t}, \tilde{t}]$.

In the first case, we include i in the set H of small subscripts of values of $|B_i^{(3)}(t)|$.

Let $i \notin H$. If $\overline{t} > \rho + \Delta$, then the control $u_i(\cdot)$ is regular on $[\overline{t}, \tilde{t}]$. Indeed, on $[\overline{t} - \Delta, \tilde{t}]$, we have $|B_i^{(3)}(t)| \geq \xi$. Moreover, on $[\overline{t} - \Delta, \tilde{t}]$, there are no instants from the set $\{t^{[1]}\}$, since they are absent on $[\rho, \eta]$. Hence, on $[\overline{t} - \Delta, \tilde{t}]$, the motion $y^{(1)}(t)$ is executed to one side of $\Pi^{c_1}(i, t)$, and, moreover, on $[\overline{t} - \Delta, \overline{t}]$, there is an instant of the discrete scheme. This implies that $u_i(\cdot)$ is regular on $[\overline{t}, \tilde{t}]$. If $\overline{t} \leq \rho + \Delta$, then, in advance, the regular control $u_i(\cdot)$ acts on $[\rho + \Delta, \tilde{t}]$, and an arbitrary control $u_i(\cdot)$ can act only on $[\overline{t}, \min\{\rho + \Delta, \tilde{t}\}]$.

Therefore, on the interval $[\overline{t}, \tilde{t}]$, the assumptions of Proposition 2.1 hold. Moreover, in estimate (5.4), we set $\omega = 0$ in the case $\overline{t} > \rho + \Delta$ and $\omega = \min\{\rho + \Delta, \tilde{t}\} - \overline{t} \leq \rho + \Delta - \overline{t}$ for $\overline{t} \leq \rho + \Delta$.

Let us sum up estimate (5.4) over the $\overline{\delta}$-intervals. We take into account that the total number of $\overline{\delta}$-intervals for each of which $H \neq \varnothing$ does not exceed $\eta - \rho$. Also, we take into account that the instant t at which $|B_i^{(3)}(t)| \geq \xi$, for an arbitrary control $u_i(t)$, $i \in \overline{1,k}$, can be only on the initial part of the interval $[\rho, \eta]$, more precisely, on $[\rho, \rho + \Delta]$. As a result, we obtain estimate (8.2). □

We now pass to the estimation of the increment of the function $V^{(2)}$ on the closed intervals of the form $E^{[h]}$, $\mathcal{E}^{[h]}$, $\mathbf{E}^{[h]}$, where $h = \overline{1, h^\flat}$.

Lemma 8.2. *The increment of the function $V^{(2)}$ on intervals of the form $E^{[h]}$, $\mathcal{E}^{[h]}$, and $\mathbf{E}^{[h]}$ is described by the following inequalities for any $h = \overline{1, h^\flat}$:*

$$\operatorname{Var}_{E^{[h]}}(V^{(2)}, [\rho, \eta]) \leq \mathbf{kf}^{[h-1]} \cdot (\eta - \rho) + e(h,k)\mathbf{sh}^{[h]} + \lambda\chi(\rho, \eta),$$
$$\operatorname{Var}_{\mathcal{E}^{[h]}}(V^{(2)}, [\rho, \eta]) \leq \mathbf{kf}^{[h]} \cdot (\eta - \rho) + \lambda\chi(\rho, \eta),$$
$$\operatorname{Var}_{\mathbf{E}^{[h]}}(V^{(2)}, [\rho, \eta]) \leq \mathbf{kf}^{[h]} \cdot (\eta - \rho) + e(h,k)\mathbf{sh}^{[h]} + \lambda\chi(\rho, \eta).$$

Proof. 1. Let $h = 1$. Using (8.1), we have

$$\operatorname{Var}_{E^{[1]}}(V^{(2)}, [\rho, \eta]) \leq \mathbf{kf} \cdot |T^{[1]}| + C_k^1\mathbf{sh}^{[1]} + \lambda\chi(T^{[1]}) + \operatorname{Var}(V^{(2)}, [\rho, \eta] \setminus T^{[1]}).$$

Here, $T^{[1]}$ is a subset of the interval $[\rho, \eta]$ filled in by loops of level 1; $|T^{[1]}|$ is the ordinary length of the set $T^{[1]}$; C_k^1 is the upper estimate of the number of loops of level 1 on $[\rho, \eta]$; $\chi(T^{[1]})$ is the interval of the form (1.9) but calculated on the set $T^{[1]}$; the summand $\operatorname{Var}(V^{(2)}, [\rho, \eta] \setminus T^{[1]})$ estimates the increment of the function $V^{(2)}$ on the set $[\rho, \eta] \setminus T^{[1]}$.

The set $[\rho, \eta] \setminus T^{[1]}$ is the set of intervals of the closed interval $[\rho, \eta]$ outside the loops of level 1. There are no more than C_k^1 such intervals, and the closure of each of these intervals is an interval of the form $\mathcal{D}$. Therefore, we can use estimate (8.2). We obtain

$$\begin{aligned}\operatorname{Var}_{E^{[1]}}(V^{(2)}, [\rho, \eta]) &\leq \mathbf{kf} \cdot |T^{[1]}| + C_k^1\mathbf{sh}^{[1]} + \mathbf{kf}^{[*]} \cdot ((\eta - \rho) - |T^{[1]}|) + C_k^1\mathbf{sh}^{[*]} + \lambda\chi(\rho, \eta) \\ &\leq \mathbf{kf}^{[0]} \cdot (\eta - \rho) + 2C_k^1\mathbf{sh}^{[1]} + \lambda\chi(\rho, \eta).\end{aligned}$$

Let us estimate $\operatorname{Var}_{\mathcal{E}^{[1]}}(V^{(2)}, [\rho, \eta])$. We take into account the relation $\mathbf{sh}^{[1]} \leq \mathbf{st}^{[1]} \cdot \mathbf{kf}$. By the definition of the set of the form $\mathcal{E}^{[1]}$, we have $\mathbf{st}^{[1]} \leq \eta - \rho$. Therefore,

$$\mathbf{sh}^{[1]} \leq \mathbf{kf} \cdot (\eta - \rho).$$

Thus,

$$\begin{aligned}\operatorname{Var}_{\mathcal{E}^{[1]}}(V^{(2)}, [\rho, \eta]) = \operatorname{Var}_{E^{[1]}}(V^{(2)}, [\rho, \eta]) &\leq \mathbf{kf}^{[0]} \cdot (\eta - \rho) + 2C_k^1\mathbf{sh}^{[1]} + \lambda\chi(\rho, \eta) \\ &\leq \mathbf{kf}^{[0]} \cdot (\eta - \rho) + 2C_k^1(\eta - \rho)\mathbf{kf} + \lambda\chi(\rho, \eta) \\ &\leq (2C_k^1 + 1)\mathbf{kf}^{[0]} \cdot (\eta - \rho) + \lambda\chi(\rho, \eta) = \mathbf{kf}^{[1]} \cdot (\eta - \rho) + \lambda\chi(\rho, \eta).\end{aligned}$$

Now let us consider the interval $[\rho, \eta]$ of the form $\mathbf{E}^{[1]}$. We can represent it as an interval composed of the initial interval $[\rho, t^{\#}]$ of the form $\mathcal{D}$, finitely many intervals of the form $\mathcal{E}^{[1]}$ following each other up to a certain instant $t^{\diamondsuit}$ (their total interval is $[t^{\#}, t^{\diamondsuit}]$), and the remaining interval $[t^{\diamondsuit}, \eta]$ of the form $E^{[1]}$. Therefore,

$$\begin{aligned}\mathrm{Var}_{\mathbf{E}^{[1]}}(V^{(2)}, [\rho, \eta]) &\le \mathrm{Var}_{\mathcal{D}}(V^{(2)}, [\rho, t^{\#}]) + \mathbf{kf}^{[1]} \cdot (t^{\diamondsuit} - t^{\#}) + \lambda\chi(t^{\#}, t^{\diamondsuit}) + \mathrm{Var}_{E^{[1]}}(V^{(2)}, [t^{\diamondsuit}, \eta]) \\ &\le \mathbf{kf}^{[*]} \cdot (t^{\#} - \rho) + \mathbf{sh}^{[*]} + \mathbf{kf}^{[1]} \cdot (t^{\diamondsuit} - t^{\#}) + \mathbf{kf}^{[0]} \cdot (\eta - t^{\diamondsuit}) + 2C_k^1 \mathbf{sh}^{[1]} + \lambda\chi(\rho, \eta) \\ &\le \mathbf{kf}^{[1]} \cdot (\eta - \rho) + (2C_k^1 + 1)\mathbf{sh}^{[1]} + \lambda\chi(\rho, \eta).\end{aligned}$$

2. Let us pass to the estimation of the increment $V^{(2)}$ on intervals of the form $E^{[h]}$, $\mathcal{E}^{[h]}$, and $\mathbf{E}^{[h]}$ for $h \in \overline{2, h^\flat}$. We prove the required inequalities by induction.

Assume that for intervals of the form $\mathbf{E}^{[p]}$, the following estimate of the increment of the function $V^{(2)}$ is obtained for $p \ge 1$, $p \le h < h^\flat$:

$$\mathrm{Var}_{\mathbf{E}^{[p]}}(V^{(2)}, [\rho, \eta]) \le \mathbf{kf}^{[p]} \cdot (\eta - \rho) + e(p, k)\mathbf{sh}^{[p]} + \lambda\chi(\rho, \eta). \tag{8.3}$$

For the interval of the form $E^{[h+1]}$, using (8.1) we have

$$\mathrm{Var}_{E^{[h+1]}}(V^{(2)}, [\rho, \eta]) \le \mathbf{kf} \cdot |T^{[h+1]}| + C_k^{h+1}\mathbf{sh}^{[h+1]} + \lambda\chi(T^{[h+1]}) + \mathrm{Var}(V^{(2)}, [\rho, \eta] \setminus T^{[h+1]}),$$

where $T^{[h+1]}$ is a subset of the interval $[\rho, \eta]$ filled in by loops of level $h+1$; $|T^{[h+1]}|$ is the length of the set $T^{[h+1]}$; C_k^{h+1} is the upper estimate of the number of loops of level $h+1$ on $[\rho, \eta]$; $\chi(T^{[h+1]})$ is the integral of the form (1.9) but calculated over the set $T^{[h+1]}$; $\mathrm{Var}(V^{(2)}, [\rho, \eta] \setminus T^{[h+1]})$ is the increment of the function $V^{(2)}$ on the set $[\rho, \eta] \setminus T^{[h+1]}$.

The set $[\rho, \eta] \setminus T^{[h+1]}$ is the set of intervals of the closed interval $[\rho, \eta]$ outside the loops of level $h+1$. There are no more than C_k^{h+1} such intervals, the closure of each of which is an interval of the form $\mathcal{D}$ or $\mathbf{E}^{[p]}$, where $p \le h$. Therefore, we can use estimates (8.2) and (8.3). We obtain

$$\begin{gathered}\mathrm{Var}_{E^{[h+1]}}(V^{(2)}, [\rho, \eta]) \le \mathbf{kf} \cdot |T^{[h+1]}| + C_k^{h+1}\mathbf{sh}^{[h+1]} \\ + \mathbf{kf}^{[h]} \cdot ((\eta - \rho) - |T^{[h+1]}|) + C_k^{h+1} e(h, k)\mathbf{sh}^{[h]} + \lambda\chi(\rho, \eta) \\ \le \mathbf{kf}^{[h]} \cdot (\eta - \rho) + C_k^{h+1}(1 + e(h, k))\mathbf{sh}^{[h+1]} + \lambda\chi(\rho, \eta).\end{gathered}$$

Let us estimate $\mathrm{Var}_{\mathcal{E}^{[h+1]}}(V^{(2)}, [\rho, \eta])$. We take into account the relation $\mathbf{sh}^{[h+1]} \le \mathbf{st}^{[h+1]} \cdot \mathbf{kf}$. By the definition of the set of the form $\mathcal{E}^{[h+1]}$, we have $\mathbf{st}^{[h+1]} \le \eta - \rho$. Therefore,

$$\mathbf{sh}^{[h+1]} \le \mathbf{kf} \cdot (\eta - \rho).$$

Thus,

$$\begin{gathered}\mathrm{Var}_{\mathcal{E}^{[h+1]}}(V^{(2)}, [\rho, \eta]) = \mathrm{Var}_{E^{[h+1]}}(V^{(2)}, [\rho, \eta]) \\ \le \mathbf{kf}^{[h]} \cdot (\eta - \rho) + C_k^{h+1}(1 + e(h, k))\mathbf{sh}^{[h+1]} + \lambda\chi(\rho, \eta) \\ \le \mathbf{kf}^{[h]} \cdot (\eta - \rho) + C_k^{h+1}(1 + e(h, k))(\eta - \rho)\mathbf{kf} + \lambda\chi(\rho, \eta) \\ \le e(h, k)\mathbf{kf}^{[0]} \cdot (\eta - \rho) + C_k^{h+1}(1 + e(h, k))\mathbf{kf}^{[0]} \cdot (\eta - \rho) + \lambda\chi(\rho, \eta).\end{gathered}$$

Since

$$\mathbf{kf}^{[h+1]} = (e(h, k) + C_k^{h+1}(1 + e(h, k)))\mathbf{kf}^{[0]},$$

it follows that

$$\mathrm{Var}_{\mathcal{E}^{[h+1]}}(V^{(2)}, [\rho, \eta]) \le \mathbf{kf}^{[h+1]} \cdot (\eta - \rho) + \lambda\chi(\rho, \eta).$$

Now let us consider the interval $[\rho, \eta]$ of the form $\mathbf{E}^{[h+1]}$. It can be represented as an interval composed of the initial interval $[\rho, t^{\#}]$ of the form $\mathbf{E}^{[h]}$, finitely many intervals of the form $\mathcal{E}^{[h+1]}$ following each other up to a certain instant $t^{\diamondsuit}$ (its total interval is $[\rho, t^{\diamondsuit}]$), and the remaining interval $[t^{\diamondsuit}, \eta]$ of the form $E^{[h+1]}$. Therefore,

$$\begin{aligned}
\mathrm{Var}_{\mathbf{E}^{[h+1]}}(V^{(2)}, [\rho, \eta]) &\le \mathrm{Var}_{\mathbf{E}^{[h]}}(V^{(2)}, [\rho, t^{\#}]) + \mathbf{kf}^{[h+1]} \cdot (t^{\diamondsuit} - \rho) \\
&\quad + \lambda\chi(\rho, t^{\diamondsuit}) + \mathrm{Var}_{E^{[h+1]}}(V^{(2)}, [t^{\diamondsuit}, \eta]) \\
&\le \mathbf{kf}^{[h]} \cdot (t^{\#} - \rho) + e(h,k)\mathbf{sh}^{[h]} + \mathbf{kf}^{[h+1]} \cdot (t^{\diamondsuit} - \rho) + \mathbf{kf}^{[h]} \cdot (\eta - t^{\diamondsuit}) \\
&\quad + C_k^{h+1}(1 + e(h,k))\mathbf{sh}^{[h+1]} + \lambda\chi(\rho, \eta) \\
&\le \mathbf{kf}^{[h+1]} \cdot (\eta - \rho) + [e(h,k) + C_k^{h+1}(1 + e(h,k))]\mathbf{sh}^{[h+1]} + \lambda\chi(\rho, \eta).
\end{aligned}$$

As a result,

$$\mathrm{Var}_{\mathbf{E}^{[h+1]}}(V^{(2)}, [\rho, \eta]) \le \mathbf{kf}^{[h+1]} \cdot (\eta - \rho) + e(h+1,k)\mathbf{sh}^{[h+1]} + \lambda\chi(\rho, \eta).$$

The lemma is proved. □

8.4. Increment of the function $V^{(2)}$ on the whole interval of the game. Recall that $h^{\flat}$ is the maximum number of h-loops along the motion $y^{(1)}(\cdot)$ on the whole interval $[t_0, \vartheta]$. Since $[t_0, \vartheta]$ is an interval of the form $\mathbf{E}^{[h^{\flat}]}$, we can write the following estimate for $\mathrm{Var}_{\mathbf{E}^{[h^{\flat}]}}(V^{(2)}, [t_0, \vartheta])$:

$$\mathrm{Var}_{\mathbf{E}^{[h^{\flat}]}}(V^{(2)}, [t_0, \vartheta]) \le \mathbf{kf}^{[h^{\flat}]} \cdot (\vartheta - t_0) + e(h^{\flat}, k)\mathbf{sh}^{[h^{\flat}]} + \lambda\chi(t_0, \vartheta).$$

We have $\mathbf{st}^{[h^{\flat}]} \le \vartheta - \vartheta_1$. Therefore,

$$\mathbf{sh}^{[h^{\flat}]} \le \mathbf{st}^{[h^{\flat}]} \cdot \mathbf{kf} \le \mathbf{kf} \cdot (\vartheta - \vartheta_1).$$

Hence

$$\begin{aligned}
\mathrm{Var}_{\mathbf{E}^{[h^{\flat}]}}(V^{(2)}, [t_0, \vartheta]) &\le e(h^{\flat}, k)\mathbf{kf}^{[0]} \cdot (\vartheta - t_0) + e(h^{\flat}, k)\mathbf{kf} \cdot (\vartheta - \vartheta_1) + \lambda\chi(t_0, \vartheta) \\
&\le 2e(h^{\flat}, k)\mathbf{kf}^{[0]} \cdot (\vartheta - \vartheta_1) + \lambda\chi(t_0, \vartheta).
\end{aligned}$$

Weakening $e(h^{\flat}, k)$ through $e(k,k)$, we finally obtain

$$\mathrm{Var}_{\mathbf{E}^{[h^{\flat}]}}(V^{(2)}, [t_0, \vartheta]) \le 2e(k,k)\mathbf{kf}^{[0]} \cdot (\vartheta - \vartheta_1) + \lambda\chi(t_0, \vartheta).$$

Replacing $\mathrm{Var}_{\mathbf{E}^{[h^{\flat}]}}(V^{(2)}, [t_0, \vartheta])$ by $V^{(2)}(\vartheta, y^{(1)}(\vartheta)) - V^{(2)}(t_0, x_0)$ and taking into account that $\gamma^{(2)}(y^{(1)}(\vartheta)) = V^{(2)}(\vartheta, y^{(1)}(\vartheta))$, we have

$$\gamma^{(2)}(y^{(1)}(\vartheta)) \le V^{(2)}(t_0, x_0) + 2e(k,k)\mathbf{kf}^{[0]} \cdot (\vartheta - \vartheta_1) + \lambda\chi(t_0, \vartheta).$$

Since $\mathbf{kf}^{[0]}$ is the maximum of the quantities

$$\mathbf{kf} = 3\lambda\mathcal{L}\mu\xi = 3\lambda((1/2) + 8\sigma)\mu\xi, \quad \mathbf{kf}^{[*]} = 4\lambda\mu\xi,$$

it follows that $\mathbf{kf}^{[0]}$ can be estimated from above by $3\lambda((3/2) + 8\sigma)\mu\xi$. Therefore,

$$\gamma^{(2)}(y^{(1)}(\vartheta)) < V^{(2)}(t_0, x_0) + 3e(k,k)\lambda(3 + 16\sigma)\mu\xi \cdot (\vartheta - \vartheta_1) + \lambda\chi(t_0, \vartheta). \tag{8.4}$$

Taking into account the distinction between the cost functions $\gamma^{(1)}$ and $\gamma^{(2)}$, by (8.4) we obtain

$$\gamma^{(1)}(y^{(1)}(\vartheta)) < V^{(2)}(t_0, x_0) + 3e(k,k)\lambda(3 + 16\sigma)\mu\xi \cdot (\vartheta - \vartheta_1) + \lambda\chi(t_0, \vartheta) + \|\gamma^{(1)} - \gamma^{(2)}\|_{\mathcal{M}}, \tag{8.5}$$

where

$$e(k,k) = C_k^k + (C_k^k + 1)[C_k^{k-1} + (C_k^{k-1} + 1)[\ldots[C_k^1 + (C_k^1 + 1)]\ldots]].$$

8.5. Final estimate. In deducing estimate (8.5), we assume that the value of ξ is fixed. According to ξ, we find c_1 and Δ_1. The quantity c_1 determines the sets $\Pi^{c_1}(i,t)$, $i=\overline{1,k}$, $t \in [t_0, \vartheta]$, using which the multivalued function $\mathbf{U}^{c_1}$ is defined. Its single-valued selection U is used as a strategy of the first player. The strategy U applies in the discrete control scheme with the step $\Delta \le \Delta_1$. For chosen U and Δ, the motion $y^{(1)}(\cdot)$ corresponds to a certain initial position $(t_0, x_0) \in \mathcal{Y}$ and a certain control $v(\cdot)$ of the second player.

By the choice of the number ξ, we can make the second summand of the right-hand side of (8.5) arbitrarily small.

The following assertion holds.

Lemma 8.3. *Let a number $\varepsilon > 0$ be given. Choose $\xi(\varepsilon) \in (0, \sigma/(1+16\sigma)]$ such that the following inequality holds*:

$$3e(k,k)\lambda(3+16\sigma)\mu\xi(\varepsilon)(\vartheta - \vartheta_1) \le \varepsilon.$$

Further, according to the chosen $\xi(\varepsilon)$, define the number $c(\varepsilon) = c_1(\xi(\varepsilon))$ and the constraint $\Delta(\varepsilon) = \Delta_1(\xi(\varepsilon))$ on the step of the discrete control scheme according to the rule of Sec 6. Let the first player apply an arbitrary strategy U with the step $\Delta \le \Delta(\varepsilon)$, and let this strategy be a single-valued selection of the multivalued function $\mathbf{U}^{c(\varepsilon)}$ constructed by using the sets $\Pi^{c(\varepsilon)}(i,t)$, $i=\overline{1,k}$, $t \in T$. Then to any initial position $(t_0, x_0) \in \mathcal{Y}$, the first player guarantees the result

$$\Gamma^{(1)}(t_0, x_0, U, \Delta) \le V^{(2)}(t_0, x_0) + \varepsilon + \lambda\chi(t_0, \vartheta) + \|\gamma^{(1)} - \gamma^{(2)}\|_{\mathcal{M}}.$$

In Lemma 8.3, we consider the strategies defined by using c-neighborhoods of the switching surfaces $\Pi(i,t)$. To formulate the result connected with geometric r-neighborhoods, we define the number

$$r(\varepsilon) := c(\varepsilon)/\lambda.$$

Then $\Pi^{r(\varepsilon)}(i,t) \subset \Pi^{c(\varepsilon)}(i,t)$ for $i=\overline{1,k}$, $t \in T$. Hence, for any strategy U embedded in $\mathbf{U}^{r(\varepsilon)}$ and for any $(t_0, x_0) \in \mathcal{Y}$, $\Delta \le \Delta(\varepsilon)$, the following inequality holds:

$$\Gamma^{(1)}(t_0, x_0, U, \Delta) \le V^{(2)}(t_0, x_0) + \varepsilon + \lambda\chi(t_0, \vartheta) + \|\gamma^{(1)} - \gamma^{(2)}\|_{\mathcal{M}},$$

which means the assertion of Theorem 1.

Remark. In the formulation of Lemma 8.3, we speak about the choice of $c(\varepsilon)$ and $\Delta(\varepsilon)$ by using the rule of Sec. 6. Such a choice uses the continuous variations of the sets $\Pi^c(i,t)$, $i=\overline{1,k}$, and is not constructive. For an effective determination of $c(\varepsilon)$ and $\Delta(\varepsilon)$, we additionally need the characteristics of the velocity of variations in t of such sets.

9. Proof of Theorem 1 for $\beta = 0$

9.1. Analogs of inequalities (5.2) and (5.4). In the case $\beta = 0$, the functions $t \to B_i^{(3)}(t)$, $i \in I$, are constant; denote the corresponding constants by $b_i^{(3)}$. For any set $F \subset I$, the vectors $b_i^{(3)}$ are either linearly dependent or linearly independent.

Choose the parameter $\hat{\xi}$ such that for any set F, the linear independence of the vectors $b_i^{(3)}$, $i \in F$, means the linear independence with the exponent $\hat{\zeta} := \hat{\xi}^{q(F)}/2$.

Let $F \subset I$ be a certain set with linearly independent vectors $b_i^{(3)}$, $i \in F$. If $\zeta \le \hat{\zeta}$ and the vector $b_j^{(3)}$, $j \in I \setminus F$, is ζ-close to the set $b_i^{(3)}$, $i \in F$, then $b_j^{(3)} \in \mathcal{G}(\{b_i^{(3)}\}_{i \in F})$. Therefore, the set $\mathcal{H}(F, \zeta, t)$ is the same for all $\zeta \le \hat{\zeta}$ and all $t \in T$.

For an arbitrary number $\xi \le \hat{\xi}$ and a set $H \subset \mathcal{H}(F, \zeta, t)$, $\zeta = \xi^{q(F)}/2$, we write inequality (5.1) for $\beta = 0$. Let us pass to the limit as $\xi \to 0$. Instead of inequality (5.1), we obtain the inequality

$$\mathcal{V}(F, t_* + \delta, y^{(1*)}(t_* + \delta)) \le V^{(2)}(t_*, x_*) + 2\lambda\omega \sum_{i \notin F \cup H} \sigma_i \mu_i + \lambda\chi(t_*, t_* + \delta).$$

Using the relation

$$2\lambda\omega \sum_{i\notin F\cup H} \sigma_i\mu_i \le 2\lambda\sigma\mu\omega,$$

we arrive at the inequality

$$\mathcal{V}(F, t_* + \delta, y^{(1*)}(t_* + \delta)) \le V^{(2)}(t_*, x_*) + 2\lambda\sigma\mu\omega + \lambda\chi(t_*, t_* + \delta), \tag{9.1}$$

which replaces inequality (5.2) for $\beta = 0$.

An analog of inequality (5.4) is the inequality

$$V^{(2)}(t, y^{(1*)}(t)) \le V^{(2)}(t_*, x_*) + 2\lambda\sigma\mu\omega + \lambda\chi(t_*, t), \quad t \in [t_*, t^*]. \tag{9.2}$$

9.2. Choice of quantities c_h and $\mathbf{st}^{[h]}$. We denote by h^* the maximum number of linearly independent vectors b_i^3, $i \in I$.

Define an arbitrary $c_{h^*} > 0$. We set $\mathbf{st}^{[h^*]} := \vartheta - \vartheta_1$.

Let the quantities $c_h > 0$ and $\mathbf{st}^{[h]}$ be introduced for a certain $h \in \overline{2, h^*}$. We define the quantities c_{h-1} and $\mathbf{st}^{[h-1]}$.

Fix a set $F_h \subset I$ of h elements such that the vectors $b_i^{(3)}$, $i \in F_h$, are linearly independent. Distinguish an arbitrary subset F_{h-1} of $h-1$ elements and the remaining subset F_1 of a single element. Using Lemma 3.3, we choose $c_\sharp \in (0, c_h]$ such that uniformly in $t \in T$ the sets $\Pi^{c_\sharp}(F_{h-1}, t)$ and $\Pi^{c_\sharp}(F_1, t)$ outside the set $\operatorname{int}\Pi^{c_h}(F_h, t)$ are distant from each other by a finite distance in the limits of a bounded set $\mathcal{X}_* \subset \mathbb{R}^n$, which is an upper estimate of the set of states at which there can be the motions of system (1.1) emanating from the set $\mathcal{Y}$. The quantity $c_\sharp$ depends on the choice of F_h and F_{h-1} and on the value of c_h, i.e., $c_\sharp = c_\sharp(F_h, F_{h-1}, c_h)$.

Let $b(F_h, F_{h-1}, c_h, c_\sharp)$ be a uniform in $\bar{t} \in T$ lower estimate of the transition time of system (1.1) from the set $\Pi^{c_\sharp}(F_{h-1}, \bar{t}) \cap \mathcal{X}_*$ to the set $\Pi^{c_\sharp}(F_1, t)$, $t \in (\bar{t}, \vartheta]$ in the case where, at the initial instant $\bar{t}$, the system is on $(\Pi^{c_\sharp}(F_{h-1}, \bar{t}) \cap \mathcal{X}_*) \setminus \operatorname{int}\Pi^{c_h}(F_h, \bar{t})$. Such an estimate can be obtained by using the continuity property of the variation in t of the set $\Pi^{c_\sharp}(F_1, t)$, which follows from Lemma 3.2.

Now we look over all sets F_h of h elements such that the vectors $b_i^{(3)}$, $i \in F_h$, are linearly independent. For each such set, we consider all variants of partition of the set F_h into a set F_{h-1} of $h-1$ elements and a singleton F_1. Let

$$c_{h-1} := \min_{(F_h, F_{h-1})} c_\sharp(F_h, F_{h-1}, c_h), \quad \mathbf{st}^{[h-1]} := \min_{(F_h, F_{h-1})} \{\mathbf{st}^{[h]}, b(F_h, F_{h-1}, c_h, c_\sharp)\}.$$

Note that for $\beta = 0$, sequentially introducing the quantities c_h, we do not define the quantities Δ_h in parallel, as was done in the case $\beta > 0$ in Sec. 6.

9.3. Formation of loops along the motion. Consider the motion $y^{(1)}(\cdot)$ of system (1.1) from the position$(t_0, x_0) \in \mathcal{Y}$, $t_0 < \vartheta$, according to a certain strategy $U \in \mathbf{U}^{c_1}$ of the first player with step $\Delta > 0$ and a certain open-loop control $v(\cdot)$ of the second player.

By the symbol $\mathfrak{H}$ we denote the set of integers $h \in \overline{1, h^*}$, for each of which there exist a set $F \subset I$ of $q(F) = h$ elements and an instant $t \in [t_0, \vartheta]$ such that $y^{(1)}(t) \in \Pi^{c_h}(F, t)$ and the set of vectors $b_i^{(3)}$, $i \in F$, is linearly independent.

If $\mathfrak{H} = \varnothing$, then we assume that the whole interval $[t_0, \vartheta]$ is a free interval of level 1.

Assume that $\mathfrak{H} \ne \varnothing$. Let $h^\flat$ be the maximum among the numbers $\mathfrak{H}$.

Along the motion $y^{(1)}(\cdot)$, we distinguish the loops related to the entrance to the sets $\Pi^{c_h}(F, t)$, where $q(F) = h$, $h = \overline{1, h^\flat}$. Also, we define free intervals.

Let $\mathfrak{T}^{[h]}$ be the set of instants $t \in [t_0, \vartheta]$ such that $y^{(1)}(t) \in \Pi^{c_h}(F, t)$ for a certain set F with $q(F) = h$, and, moreover, let the vectors $b_i^{(3)}$, $i \in F$, be linearly independent. The above F is not necessarily unique. Denote the collection of sets F corresponding to the instant t by $\{F^{[h]}(t)\}$.

Taking into account the definition of the number h^*, we obtain that $\{F^{[h^*]}(t)\}$ is the same for any $t \in \mathfrak{T}^{[h^*]}$.

1. If $h^\flat = h^*$, then we set

$$t_1^{[h^*]} := \min\{t : t \in \mathfrak{T}^{[h^*]}\}, \quad \hat{t}^{[h^*]} := \max\{t : t \in \mathfrak{T}^{[h^*]}\}.$$

The interval $[t_1^{[h^*]}, \hat{t}^{[h^*]}]$ is called a loop of level h^*. The intervals $[t_0, t_1^{[h^*]}]$ and $[\hat{t}^{[h^*]}, \vartheta]$ are called free intervals of level h^*.

If $h^\flat < h^*$, then we assume that $[t_0, \vartheta]$ is a free interval of level $h^\flat + 1$.

2. Assume that the loops and the free intervals corresponding to a certain level $h \in \overline{2, h^\flat}$ for $h^\flat = h^*$ and to a level $h \in \overline{2, h^\flat + 1}$ for $h^\flat < h^*$ are defined. Let $\Theta_j^{[h]} := [\theta_j^{[h]}, \theta_{j+}^{[h]}]$ be the notation of the free interval with the number j in the through enumeration on $[t_0, \vartheta]$.

Consider the free interval $\Theta_j^{[h]}$. Let

$$\tau_{j,1}^{[h-1]} := \min\big\{t : t \in \mathfrak{T}^{[h-1]} \cap \Theta_j^{[h]}\big\},$$
$$\tau_{j,i+1}^{[h-1]} := \min\Big\{t : t \in \mathfrak{T}^{[h-1]} \cap [\tau_{j,i}^{[h-1]} + \mathbf{st}^{[h-1]}, \vartheta_{j+}^{[h]}]\Big\}, \quad i = 1, 2, \ldots.$$

The obtained set of instants $\tau_{j,i}^{[h-1]}$ is denoted by $\{\tau_j^{[h-1]}\}$.

With each instant $\tau_{j,i}^{[h-1]}$, we associate loops of level $h-1$. Choose an arbitrary $F \in \{F^{[h-1]}(t_{j,i}^{[h-1]})\}$. We set

$$t_{j,i}^{[h-1]} := \tau_{j,i}^{[h-1]},$$
$$t_{j,i+}^{[h-1]} := \max\Big\{t : y^{(1)}(t) \in \Pi^{c_{h-1}}(F, t),\ t \in [\tau_{j,i}^{[h-1]}, \tau_{j,i}^{[h-1]} + \mathbf{st}^{[h-1]}] \cap \Theta_j^{[h]}\Big\}.$$

Proposition 9.1. *On the interval $(t_{j,i+}^{[h-1]}, \tau_{j,i}^{[h-1]} + \mathbf{st}^{[h-1]}] \cap \Theta_j^{[h]}$, there are no instants from $\mathfrak{T}^{[h-1]}$.*

Proof. Assume the contrary. Let $\hat{t}$ be an instant from the interval considered belonging to $\mathfrak{T}^{[h-1]}$. By the symbol $\widehat{F}$ we denote an arbitrary set in $\{F^{[h-1]}(\hat{t})\}$. The vectors $b_i^{(3)}$, $i \in \widehat{F}$, are linearly independent.

First assume that among the vectors $b_i^{(3)}$, $i \in \widehat{F}$, there is at least one vector $b_{\hat{i}}^{(3)}$ not belonging to the linear span of the vectors $b_i^{(3)}$, $i \in F$. Then the vectors $b_i^{(3)}$, $i \in \widetilde{F} = F \cup \{\hat{\imath}\}$, are linearly independent. Taking into account the inclusion $[\tau_{j,i}^{[h-1]}, \theta_{j+}^{[h]}] \subset \Theta_j^{[h]}$, we obtain

$$y^{(1)}(\tau_{j,i}^{[h-1]}) \notin \operatorname{int} \Pi^{c_h}(\widetilde{F}, t).$$

Moreover,

$$y^{(1)}(\tau_{j,i}^{[h-1]}) \in \Pi^{c_{h-1}}(F, t).$$

Hence, by Lemmas 3.2 and 3.3, taking into account the choice of the numbers c_{h-1} and $\mathbf{st}^{[h-1]}$, we see that the motion $y^{(1)}(t)$ on $[\tau_{j,i}^{[h-1]}, \tau_{j,i}^{[h-1]} + \mathbf{st}^{[h-1]}] \cap \Theta_j^{[h]}$ cannot attain $\Pi^{c_{h-1}}(\hat{\imath}, t)$, and, therefore, the same also on $\Pi^{c_{h-1}}(\widehat{F}, t)$.

Now let any vector $b_i^{(3)}$, $i \in \widehat{F}$, belong to the linear span of the vectors $b_i^{(3)}$, $i \in F$. Since each of these two sets consists of $h-1$ linearly independent vectors, it follows that $\Pi^{c_{h-1}}(F, t) = \Pi^{c_{h-1}}(\widehat{F}, t)$. Hence, $y^{(1)}(\hat{t}) \in \Pi^{c_{h-1}}(F, \hat{t})$, which contradicts the definition of the instant $t_{j,i+}^{[h-1]}$. $\square$

It follows from Proposition 9.1 that with each instant $\tau_{j,i}^{[h-1]}$, we associate a unique loop of level $h-1$ but not a series of loops as in Sec. 7 for $\beta > 0$.

The instant $t_{j,i}^{[h-1]} = \tau_{j,i}^{[h-1]}$ is called the instant of origin of the loop with the number i of level $h-1$ on the free interval $\Theta_j^{[h]}$, and the instant $t_{j,i+}^{[h-1]}$ is called the instant of the end of this loop.

Passing through all free intervals of level h, we enumerate (in through numbering) the loops of level $h-1$. Let $\{t^{[h-1]}\}$ be the instants of origin of the loops of level $h-1$.

9.4. Increment of the function $V^{(2)}$. To estimate the increment of the function $V^{(2)}$ along the motion $y^{(1)}(t)$, we represent the interval $[t_0, \vartheta]$ as an interval composed of intervals of loops of levels $h = \overline{1, h^\flat}$ and free intervals of level 1.

Let $[t_*, t^*]$ be the interval of a certain loop of level h. Then, taking into account (9.1), we estimate the increment $\mathrm{Var}(V^{(2)}, [t_*, t^*])$ of the function $V^{(2)}$ as follows:

$$\mathrm{Var}(V^{(2)}, [t_*, t^*]) \le 2\lambda\sigma\mu\Delta + c_h + \lambda\chi(t_*, t^*).$$

If $[t_*, t^*]$ is a free interval of level 1, then by (9.2) we have

$$\mathrm{Var}(V^{(2)}, [t_*, t^*]) \le 2\lambda\sigma\mu\Delta + \lambda\chi(t_*, t^*).$$

Denote

$$\mathbf{sh}^{[*]} := 2\lambda\sigma\mu\Delta; \quad \mathbf{sh}^{[h]} := 2\lambda\sigma\mu\Delta + c_h, \quad h = \overline{1, h^\flat}. \tag{9.3}$$

Therefore, to estimate the increment of the function $V^{(2)}$ on the interval of a loop of level h we will use the inequality

$$\mathrm{Var}(V^{(2)}, [t_*, t^*]) \le \mathbf{sh}^{[h]} + \lambda\chi(t_*, t^*), \tag{9.4}$$

and on the free interval of level 1 we will use the inequality

$$\mathrm{Var}(V^{(2)}, [t_*, t^*]) \le \mathbf{sh}^{[*]} + \lambda\chi(t_*, t^*). \tag{9.5}$$

For each $h = \overline{1, h^\flat}$, by the symbol $a^{[h]}$ we denote the number of loops of level h and by the symbol $m^{[h]}$ the number of free intervals of level h on $[t_0, \vartheta]$. Let us agree that $m^{[h^\flat+1]} := 1$.

Taking into account (9.4) and (9.5), we estimate the increment of the function $V^{(2)}$ on $[t_0, \vartheta]$ by the inequality

$$\mathrm{Var}(V^{(2)}, [t_0, \vartheta]) \le \sum_{h=1}^{h^\flat} a^{[h]}\mathbf{sh}^{[h]} + m^{[1]}\mathbf{sh}^{[*]} + \lambda\chi(t_0, \vartheta). \tag{9.6}$$

1. Let us estimate from above the possible number of loops and free intervals.

If $h^\flat = h^*$, then

$$a^{[h^\flat]} = 1, \quad m^{[h^\flat]} \le 2. \tag{9.7}$$

In the case $h^\flat < h^*$, the following estimates hold:

$$a^{[h^\flat]} \le [\![(\vartheta - t_0)/\mathbf{st}^{[h^\flat]}]\!] + 1, \quad m^{[h^\flat]} \le a^{[h^\flat]} + 1. \tag{9.8}$$

Here, the double square brackets stand for the integral part.

For $h \in \overline{1, (h^\flat - 1)}$, we have

$$a^{[h]} \le [\![(\vartheta - t_0)/\mathbf{st}^{[h]}]\!] + m^{[h+1]}, \quad m^{[h]} \le a^{[h]} + m^{[h+1]}. \tag{9.9}$$

A. Let us show that for any $h \in \overline{1, h^\flat}$, the following inequality holds:

$$m^{[h]} \le \sum_{p=1}^{h^\flat - h + 1} 2^{(p-1)} [\![(\vartheta - t_0)/\mathbf{st}^{[h+p-1]}]\!] + 2^{(h^\flat - h + 1)}. \tag{9.10}$$

If $h = h^\flat$, then the above inequality holds. Assume that for $h+1$, where $h \in \overline{1, (h^\flat - 1)}$, inequality (9.10) is proved, i.e.,

$$m^{[h+1]} \le \sum_{p=1}^{h^\flat - h} 2^{(p-1)} [\![(\vartheta - t_0)/\mathbf{st}^{[h+p]}]\!] + 2^{(h^\flat - h)}.$$

Then using (9.7)–(9.9), we obtain

$$m^{[h]} \le a^{[h]} + m^{[h+1]} \le [\![(\vartheta - t_0)/\mathbf{st}^{[h]}]\!] + 2m^{[h+1]}$$

$$\le [\![(\vartheta - t_0)/\mathbf{st}^{[h]}]\!] + \sum_{p=1}^{h^\flat - h} 2^p [\![(\vartheta - t_0)/\mathbf{st}^{[h+p]}]\!] + 2^{(h^\flat - h+1)}$$

$$= \sum_{p=1}^{h^\flat - h+1} 2^{(p-1)} [\![(\vartheta - t_0)/\mathbf{st}^{[h+p-1]}]\!] + 2^{(h^\flat - h+1)}.$$

Therefore, (9.10) is proved.

Taking into account that $\mathbf{st}^{[h]} \le \mathbf{st}^{[h+1]} \le \cdots \le \mathbf{st}^{[h^\flat]}$, we deduce from inequality (9.10) that

$$m^{[h]} \le \left(\sum_{p=1}^{h^\flat - h+1} 2^{(p-1)} \right) [\![(\vartheta - t_0)/\mathbf{st}^{[h]}]\!] + 2^{(h^\flat - h+1)} = (2^{(h^\flat - h+1)} - 1)[\![(\vartheta - t_0)/\mathbf{st}^{[h]}]\!] + 2^{(h^\flat - h+1)}. \quad (9.11)$$

In particular, for $h = 1$, we have

$$m^{[1]} \le (2^{(h^\flat)} - 1)[\![(\vartheta - t_0)/\mathbf{st}^{[1]}]\!] + 2^{h^\flat}. \quad (9.12)$$

B. Using (9.7)–(9.9), we write the inequality

$$\sum_{h=1}^{h^\flat} a^{[h]} \le \sum_{h=1}^{h^\flat} [\![(\vartheta - t_0)/\mathbf{st}^{[h]}]\!] + \sum_{h=1}^{h^\flat} m^{[h+1]}. \quad (9.13)$$

Using (9.11), we obtain

$$\sum_{h=1}^{h^\flat} m^{[h+1]} = \sum_{h=2}^{h^\flat} m^{[h]} + 1 \le \sum_{h=2}^{h^\flat} (2^{(h^\flat - h+1)} - 1)[\![(\vartheta - t_0)/\mathbf{st}^{[h]}]\!] + \sum_{h=2}^{h^\flat} 2^{(h^\flat - h+1)} + 1.$$

Substituting the estimate for $\sum\limits_{h=1}^{h^\flat} m^{[h+1]}$ in (9.13), we obtain the inequalities

$$\sum_{h=1}^{h^\flat} a^{[h]} \le [\![(\vartheta - t_0)/\mathbf{st}^{[1]}]\!] + \sum_{h=2}^{h^\flat} 2^{(h^\flat - h+1)} [\![(\vartheta - t_0)/\mathbf{st}^{[h]}]\!] + \sum_{h=2}^{h^\flat} 2^{(h^\flat - h+1)} + 1$$

$$\le [\![(\vartheta - t_0)/\mathbf{st}^{[1]}]\!] + \left(\sum_{h=2}^{h^\flat} 2^{(h^\flat - h+1)} \right) [\![(\vartheta - t_0)/\mathbf{st}^{[2]}]\!] + \sum_{h=2}^{h^\flat} 2^{(h^\flat - h+1)} + 1.$$

Since

$$\sum_{h=2}^{h^\flat} 2^{(h^\flat - h+1)} = 2^{h^\flat} - 2$$

and $\mathbf{st}^{[1]} \le \mathbf{st}^{[2]}$, it follows that

$$\sum_{h=1}^{h^\flat} a^{[h]} \le [\![(\vartheta - t_0)/\mathbf{st}^{[1]}]\!] + 2^{h^\flat} [\![(\vartheta - t_0)/\mathbf{st}^{[1]}]\!] - 2[\![(\vartheta - t_0)/\mathbf{st}^{[1]}]\!] + 2^{h^\flat} - 1$$

$$= (2^{h^\flat} - 1)[\![(\vartheta - t_0)/\mathbf{st}^{[1]}]\!] + 2^{h^\flat} - 1.$$

Therefore,

$$\sum_{h=1}^{h^\flat} a^{[h]} \le (2^{h^\flat} - 1)[\![(\vartheta - t_0)/\mathbf{st}^{[1]}]\!] + 2^{h^\flat} - 1. \quad (9.14)$$

2. For brevity, denote

$$\varkappa^\flat := (2^{h^\flat} - 1)[\![(\vartheta - t_0)/\mathbf{st}^{[1]}]\!] + 2^{h^\flat}.$$

Then by (9.3), (9.6), (9.12), and (9.14), we obtain

$$\mathrm{Var}(V^{(2)},[t_0,\vartheta]) \le (\varkappa^\flat-1)\mathbf{sh}^{[h^\flat]}+\varkappa^\flat\mathbf{sh}^{[*]}+\lambda\chi(t_0,\vartheta)=(2\varkappa^\flat-1)2\lambda\sigma\mu\Delta+(\varkappa^\flat-1)c_{h^\flat}+\lambda\chi(t_0,\vartheta).$$

The number $h^\flat$ is related to the motion $y^{(1)}(\cdot)$. To exclude such a dependence, we estimate $h^\flat$ through h^* and estimate $c_{h^\flat}$ through c_{h^*}. We have

$$\mathrm{Var}(V^{(2)},[t_0,\vartheta]) \le (2\varkappa^*-1)2\lambda\sigma\mu\Delta+(\varkappa^*-1)c_{h^*}+\lambda\chi(t_0,\vartheta), \tag{9.15}$$

where

$$\varkappa^* := (2^{h^*}-1)[\![(\vartheta-t_0)/\mathbf{st}^{[1]}]\!]+2^{h^*}.$$

Using (9.15), we obtain the inequality

$$\gamma^{(1)}(y^{(1)}(\vartheta)) \le V^{(2)}(t_0,x_0)+(2\varkappa^*-1)2\lambda\sigma\mu\Delta+(\varkappa^*-1)c_{h^*}+\lambda\chi(t_0,\vartheta)+\|\gamma^{(1)}-\gamma^{(2)}\|_{\mathcal{M}}. \tag{9.16}$$

The quantity $\mathbf{st}^{[1]}$ on which $\varkappa^*$ depends is determined by $c_1,c_2,\dots,c_{h^*}$ and does not increase when they decrease. Therefore, the second and third summands on the right-hand side tend to zero as $\Delta\to 0$, $c_{h^*}\to 0$. On the whole, estimate (9.16) is not constructive, since, in a nonconstructive way, by using c_{h^*}, the sequence $c_{h^*-1},\dots,c_2,c_1$ is defined.

Estimate (9.16) implies the assertion of Theorem 1 with $\beta=0$.

10. Case of the Scalar Control of the First Player

Theorem 1 holds for the general case $k\ge 1$. In the scalar case $k=1$, the result can be strengthened.

10.1. Constructions arising in the scalar case. For $q(F)=k$, i.e., when $F=I$, the formulation of the main Lemma 2.1 becomes substantially simplified. The set H becomes empty, assumption (2.2) becomes extra, and this is the case for the assumption on regular controls for subscripts $i\in I\setminus(F\cup H)$. In estimate (2.3), the third and fourth summands on the right-hand side disappear. The formulation of the lemma takes the following form.

Proposition 10.1. *Assume that $F=I$ and $(t_*,x_*)\in Z,\delta>0,t_*+\delta\le\vartheta$. Let $y^{(1*)}(\cdot)$ be the motion of system* (1.1) *according to admissible open-loop controls $u(\cdot)$ and $v(\cdot)$ emanating from the point x_* at the instant t_*. Then the following estimate holds*:

$$\mathcal{V}(F,t_*+\delta,y^{(1*)}(t_*+\delta)) \le V^{(2)}(t_*,x_*)+\lambda\delta^2\sum_{i=1}^{k}\beta_i\mu_i+\lambda\chi(t_*,t_*+\delta).$$

Now rewrite Proposition 2.1 for the case $F=\varnothing$ and $H=\varnothing$.

Proposition 10.2. *Let $(t_*,x_*)\in Z,t^*\in(t_*,\vartheta]$. Let $0\le\omega\le t^*-t_*$, and along the motion $y^{(1*)}(\cdot)$, emanating from the point x_* at the instant t_*, for any $i\in I$, let either $y^{(1*)}(t)\in\Pi_+(i,t)$ on the interval $[t_*+\omega,t^*]$, and, moreover, let $u_i(t)=\mu_i$, or let $y^{(1*)}(t)\in\Pi_-(i,t)$, and, moreover, let $u_i(t)=-\mu_i$. Then the following estimate holds for any $t\in[t_*,t^*]$*:

$$V^{(2)}(t,y^{(1*)}(t)) \le V^{(2)}(t_*,x_*)+2\lambda\omega\sum_{i=1}^{k}\sigma_i\mu_i+\lambda\chi(t_*,t).$$

We will use Propositions 10.1 and 10.2 in the case of scalar control of the first player, i.e., for $k=1$.

Therefore, we assume that the control action of the first player is subjected to the constraint $|u|\le\mu$ and the matrices $B^{(1)}(t)$ and $B^{(2)}(t)$ are columns of dimension n. For each t, we deal with a single switching surface; omitting the symbol F, we will denote it by $\Pi(t)$. Analogously, we will omit the symbol F in the notation of an r-neighborhood of the surface $\Pi(t)$ and also in that of parts of the space $\mathbb{R}^n$ that are determined by these sets.

In the scalar case, Condition 2 holds automatically. There is no necessity to consider c-neighborhoods of switching surfaces, and Lemma 3.3 becomes extra. Also, we may not introduce the smallness parameter

ξ. There arises a possibility of obtaining an explicit estimate for the result, which is guaranteed to the first player by the strategy based on the switching surfaces $\Pi(t)$ being applied with an arbitrary step $\Delta > 0$ of the discrete control scheme.

10.2. Proof of Theorem 2. Fix a number $r \geq 0$. Consider the motion $y^{(1)}(\cdot)$ of system (1.1) from the position $(t_0, x_0) \in \mathcal{Y}$, $t_0 < \vartheta$, according to a certain strategy $U \in \mathbf{U}^r$ of the first player with a step Δ of the discrete control scheme and certain $v(\cdot) \in K^{(1)}$.

1. Let $\beta > 0$. We set

$$\mathbf{st} := \sqrt{\frac{2\sigma\mu\Delta + r}{\beta\mu}}. \tag{10.1}$$

A. Along the motion $y^{(1)}(\cdot)$, we distinguish loops related to the entrance to the sets $\Pi^r(t)$. Also, we define free intervals.

We set

$$t_1 := \min\bigl\{t : y^{(1)}(t) \in \Pi^r(t),\ t \in [t_0, \vartheta]\bigr\}.$$

The instant t_1 is called the instant of origin of the first loop. Further, we distinguish the instant t_{1+} of the end of the first loop:

$$t_{1+} := \max\bigl\{t : y^{(1)}(t) \in \Pi^r(t),\ t \in [t_1, t_1 + \mathbf{st}] \cap [t_0, \vartheta]\bigr\}.$$

In particular, the instant t_{1+} can coincide with t_1.

As the instant t_2 of the origin of the first loop, we take the instant

$$t_2 := \min\bigl\{t : y^{(1)}(t) \in \Pi^r(t),\ t \in [t_1 + \mathbf{st},\ \vartheta]\bigr\}.$$

Then we distinguish the instant t_{2+} of the end of the second loop:

$$t_{2+} := \max\bigl\{t : y^{(1)}(t) \in \Pi^r(t),\ t \in [t_2, t_2 + \mathbf{st}] \cap [t_0, \vartheta]\bigr\}.$$

Continuing this process, we obtain the set of loops on $[t_0, \vartheta]$.

From $[t_0, \vartheta]$, we remove the interiors of intervals of the constructed loops. We obtain an ordered set of closed intervals. Each of them is called a free interval. A free interval can degenerate, i.e., it can be a singleton.

If there are no loops on $[t_0, \vartheta]$, then we assume that $[t_0, \vartheta]$ is a free interval.

B. Let $[\rho, \eta]$ be a certain free interval. We show that the increment of the function $V^{(2)}$ on it is described by the inequality

$$\mathrm{Var}_{\mathcal{D}}(V^{(2)}, [\rho, \eta]) \leq 2\lambda\sigma\mu\Delta + \lambda\chi(\rho, \eta). \tag{10.2}$$

Here, the subscript $\mathcal{D}$ stresses the property that the variation of the function $V^{(2)}$ is calculated on the free interval.

Along the motion $y^{(1)}(\cdot)$, a certain control $u(\cdot)$ is realized. The value of $u(t)$ is said to be "regular" if $u(t) = \mu$ ($u(t) = -\mu$) for $y^{(1)}(t) \in \Pi_+(t)$ ($y^{(1)}(t) \in \Pi_-(t)$).

On the interior of a free interval, the point $y^{(1)}(t)$ is outside the set $\Pi^r(t)$ and, therefore, does not attain the set $\Pi(t)$. Therefore, for $\Delta \leq \eta - \rho$, the control $u(t)$ is regular on $[\rho + \Delta, \eta)$ and can be arbitrary only on $[\rho, \rho + \Delta)$. Estimate (10.2) directly follows from Proposition 10.2 for $\omega = \Delta$, $t_* = \rho$, and $t^* = \eta$.

If $\Delta > \eta - \rho$, then we apply Proposition 10.2 for $\omega = \eta - \rho$, $t_* = \rho$, and $t^* = \eta$. Once again, we obtain estimate (10.2).

C. We say that $[\rho, \eta]$ is an interval of the form E if it is composed of a certain loop $[t_i, t_{i+}]$ and a free interval adjacent to it to the right. An interval $[\rho, \eta]$ of the form E satisfying the additional condition $\rho + \mathbf{st} \leq \eta$ is called an interval of the form $\mathcal{E}$.

Let us estimate the increment of the function $V^{(2)}$ along the motion $y^{(1)}(\cdot)$ on the interval of the form E.

Consider the interval of the loop $[t_i, t_{i+}]$. Applying Proposition 10.1 for $\delta = t_{i+} - t_i$, we have

$$\mathcal{V}(t_{i+}, y^{(1)}(t_{i+})) \leq V^{(2)}(t_i, y^{(1)}(t_i)) + \lambda\beta\mu(t_{i+} - t_i)^2 + \lambda\chi(t_i, t_{i+}).$$

Since $t_{i+} - t_i \le \mathbf{st}$, it follows that

$$\mathcal{V}(t_{i+}, y^{(1)}(t_{i+})) \le V^{(2)}(t_i, y^{(1)}(t_i)) + \lambda\beta\mu\mathbf{st} \cdot (t_{i+} - t_i) + \lambda\chi(t_i, t_{i+}).$$

Taking into account the inequality

$$V^{(2)}(t_{i+}, y^{(1)}(t_{i+})) \le \mathcal{V}(t_{i+}, y^{(1)}(t_{i+})) + \lambda r,$$

we arrive at the relation

$$\mathrm{Var}(V^{(2)}, [t_i, t_{i+}]) \le \lambda\beta\mu\mathbf{st} \cdot (t_{i+} - t_i) + \lambda r + \lambda\chi(t_i, t_{i+}). \tag{10.3}$$

On the free interval $[t_{i+}, \eta]$, by (10.2) we have

$$\mathrm{Var}_{\mathcal{D}}(V^{(2)}, [t_{i+}, \eta]) \le 2\lambda\sigma\mu\Delta + \lambda\chi(t_{i+}, \eta). \tag{10.4}$$

Combining (10.3) and (10.4) and taking into account the inequality $t_{i+} - t_i \le \eta - \rho$, we obtain

$$\mathrm{Var}_E(V^{(2)}, [\rho, \eta]) \le \lambda\beta\mu\mathbf{st} \cdot (\eta - \rho) + 2\lambda\sigma\mu\Delta + \lambda r + \lambda\chi(\rho, \eta). \tag{10.5}$$

The subscript E stresses the property that the calculation of the increment of the function $V^{(2)}$ is performed on the interval of the form E.

We now pass to the estimation of the increment of the function $V^{(2)}$ along the motion $y^{(1)}(\cdot)$ on an interval of the form $\mathcal{E}$. Since $\eta - \rho \ge \mathbf{st}$ in this case, (10.1) implies the inequality

$$2\lambda\sigma\mu\Delta + \lambda r \le \lambda\beta\mu\mathbf{st} \cdot (\eta - \rho).$$

Taking into account (10.5), we obtain

$$\mathrm{Var}_{\mathcal{E}}(V^{(2)}, [\rho, \eta]) \le 2\lambda\beta\mu\mathbf{st} \cdot (\eta - \rho) + \lambda\chi(\rho, \eta). \tag{10.6}$$

D. Consider the interval $[t_0, \vartheta]$. Represent it as an interval composed of the first free interval $[t_0, t^{\#}]$, finitely many intervals of the form $\mathcal{E}$ following each other from the instant $t^{\#}$ up to a certain instant $t^{\diamondsuit}$ (their total interval is $[t^{\#}, t^{\diamondsuit}]$), and the remaining interval $[t^{\diamondsuit}, \vartheta]$ of the form E. Sequentially applying estimates (10.2), (10.6), and (10.5), we have

$$\begin{gathered}
\mathrm{Var}(V^{(2)}, [t_0, \vartheta]) = \mathrm{Var}_{\mathcal{D}}(V^{(2)}, [t_0, t^{\#}]) + \mathrm{Var}(V^{(2)}, [t^{\#}, t^{\diamondsuit}]) \\
+ \mathrm{Var}_E(V^{(2)}, [t^{\diamondsuit}, \vartheta]) \le 2\lambda\sigma\mu\Delta + 2\lambda\beta\mu\mathbf{st} \cdot (t^{\diamondsuit} - t^{\#}) + \lambda\beta\mu\mathbf{st} \cdot (\vartheta - t^{\diamondsuit}) + 2\lambda\sigma\mu\Delta + \lambda r + \lambda\chi(t_0, \vartheta) \\
\le 2\lambda\beta\mu\mathbf{st} \cdot (\vartheta - t_0) + 4\lambda\sigma\mu\Delta + \lambda r + \lambda\chi(t_0, \vartheta).
\end{gathered}$$

Substituting $\mathbf{st}$, by formula (10.1) we obtain

$$\mathrm{Var}(V^{(2)}, [t_0, \vartheta]) \le 2\lambda\sqrt{(2\sigma\mu\Delta + r)\beta\mu}(\vartheta - t_0) + 4\lambda\sigma\mu\Delta + \lambda r + \lambda\chi(t_0, \vartheta). \tag{10.7}$$

2. Let $\beta = 0$. We set

$$t_1 := \min\big\{t : y^{(1)}(t) \in \Pi^r(t),\ t \in [t_0, \vartheta]\big\}, \quad \hat{t} := \max\big\{t : y^{(1)}(t) \in \Pi^r(t),\ t \in [t_0, \vartheta]\big\}.$$

We have

$$y^{(1)}(t) \notin \Pi^r(t), \quad t \in [t_0, t_1) \cup (\hat{t}, \vartheta].$$

For the intervals $[t_0, t_1]$ and $[\hat{t}, \vartheta]$, using Proposition 10.2 (as in deducing inequality (10.2)), we obtain

$$\mathrm{Var}(V^{(2)}, [t_0, t_1]) \le 2\lambda\sigma\mu\Delta + \lambda\chi(t_0, t_1), \tag{10.8}$$

$$\mathrm{Var}(V^{(2)}, [\hat{t}, \vartheta]) \le 2\lambda\sigma\mu\Delta + \lambda\chi(\hat{t}, \vartheta). \tag{10.9}$$

For the interval $[t_1, \hat{t}]$, using Proposition 10.1, for $\beta = 0$ we have

$$\mathcal{V}(\hat{t}, y^{(1)}(\hat{t})) \le V^{(2)}(t_1, y^{(1)}(t_1)) + \lambda\chi(t_1, \hat{t}),$$

and, therefore, taking into account the inequality

$$V^{(2)}(\hat{t}, y^{(1)}(\hat{t})) \le \mathcal{V}(\hat{t}, y^{(1)}(\hat{t})) + \lambda r,$$

we arrive at the estimate

$$\mathrm{Var}(V^{(2)}, [t_1, \hat{t}]) \le \lambda r + \lambda\chi(t_1, \hat{t}). \tag{10.10}$$

Combining (10.8)–(10.10), we obtain

$$\mathrm{Var}(V^{(2)}, [t_0, \vartheta]) \le 4\lambda\sigma\mu\Delta + \lambda r + \lambda\chi(t_0, \vartheta). \tag{10.11}$$

3. Using (10.7) in the case $\beta > 0$ and (10.11) in the case $\beta = 0$, we obtain the estimate

$$V^{(2)}(\vartheta, y^{(1)}(\vartheta)) \le V^{(2)}(t_0, x_0) + 2\lambda\sqrt{(2\sigma\mu\Delta + r)\beta\mu}(\vartheta - t_0) + 4\lambda\sigma\mu\Delta + \lambda r + \lambda\chi(t_0, \vartheta). \tag{10.12}$$

Since

$$\gamma^{(2)}(y^{(1)}(\vartheta)) = V^{(2)}(\vartheta, y^{(1)}(\vartheta)), \quad \gamma^{(1)}(y^{(1)}(\vartheta)) \le \gamma^{(2)}(y^{(1)}(\vartheta)) + \|\gamma^{(1)} - \gamma^{(2)}\|_{\mathcal{M}}$$

and the right-hand side of (10.12) is independent of the chosen $v(\cdot) \in K^{(1)}$, it follows that

$$\Gamma^{(1)}(t_0, x_0, U, \Delta) \le V^{(2)}(t_0, x_0) + 2\lambda\sqrt{(2\sigma\mu\Delta + r)\beta\mu}(\vartheta - t_0)$$
$$+4\lambda\sigma\mu\Delta + \lambda r + \lambda\chi(t_0, \vartheta) + \|\gamma^{(1)} - \gamma^{(2)}\|_{\mathcal{M}},$$

which completes the proof of Theorem 2. □

11. Tests of Numerical Construction of Switching Surfaces in Differential Games

In this paper, we do not discuss algorithms for numerically constructing switching surfaces. We restrict ourselves to a brief description of publications in which the results of computer modeling with the use of switching surfaces are presented.

The simplest case is the case $n = 2$, i.e., when the values of the cost function at the instant of termination of the game are determined by certain two coordinates of the state vector.

For this case, at the Institute of Mathematics and Mechanics of the Ural Department of the Russian Academy of Sciences, in the early 1980s, effective algorithms (see [1, 3, 11, 16, 18, 21, 35]) for constructing t-sections of the level sets of the value function were elaborated. The constructions are performed in the framework of the approximating game (1.4) on a given grid $\{t_j\}$ of instants of time and on a certain grid $\{c_p\}$ of values of the value function. Each section $W^{(2)}_{c_p}(t_j)$ is a convex polygon on the plane. The passage from the constructed section $W^{(2)}_{c_p}(t_j)$ to the section $W^{(2)}_{c_p}(t_{j-1})$, $t_{j-1} < t_j$, is performed by using a special backward procedure that uses the convexification operation of a positively homogeneous piecewise-linear function in the space $\mathbb{R}^2$ or the operation of intersection of broken lines on the plane, which is equivalent to it.

An easy processing (see [3, 4, 9, 10, 18]) of polygons $W^{(2)}_{c_p}(t_j)$, $c \in \{c_p\}$, for each component u_i, $i = \overline{1, k}$, of the control action u of the first player yields the switching lines corresponding to the instant t_j. The switching lines calculated on the grid $\{t_j\}$ determine the control method with respect to the component u_i. The sets of switching lines are stored in the memory and are used in the discrete control scheme.

In [3–7, 12–15, 17–19, 25, 26, 38], the problem on the landing of an aircraft in wind shear is considered. The process of landing up to the flight overshooting the end of the runway is studied. Under the linearization of nonlinear dynamical equations with respect to the nominal motion along the rectilinear trajectory of descent, we obtain a linear system breaking up into two subsystems; one of them contains state variables and control actions that influence the vertical deviation from the nominal (subsystem of longitudinal channel), and the other contains those that influence the lateral deviation (subsystem of lateral channel). Respectively, two auxiliary differential games with a fixed instant of termination are considered. In the first of them, the cost function depends on the vertical deviation at the instant of termination and on the speed of its variation. The control actions are the deviation of the elevator and the variation of the traction force. In the second game, the cost is determined by the lateral deviation and its speed. The control action are the deviations of ailerons and direction rudder.

For each of the auxiliary linear differential games and for a given grid of instants of time, the switching lines defining the control action of the first player close to optimal (in the framework of a linear model) are calculated. The testing was performed within the framework of the initial nonlinear system. Different variants of wind disturbance are applied: the formation of the disturbance by the feedback principle on the basis of solution of auxiliary linear differential games [6, 12, 14, 15], models of wind microbursts [22, 33, 51], and the formation of the disturbance by using a random number generator. The results of modeling of the landing process are compared with those given by the traditional control methods.

The control method using sets of switching lines was also tested by examining model landing and launching problems, which were proposed by Miele and his collaborators in [32–34]. The results of the study are contained in [15, 23, 43–45]. For the launching problem, a comparison with the results of G. Leitmann and his collaborators presented in [20, 31], in which the control is constructed by using appropriate Lyapunov functions, was carried out.

In [8], the problem of the take-off run of an aircraft along a runway under a wind perturbation is considered. The control method studied is based on the construction of switching lines.

In [39, 40], the problem of transportation of a load with movable suspension point is considered in the game-theoretic statement. The switching lines defining the optimal control method are constructed.

A software package for constructing switching surfaces in the case $n = 3$ is described in [48].

REFERENCES

1. N. D. Botkin, "Numerical construction of sections of the positional consumption set in a linear differential game," in: *Collection* (*Materials on Computer Software*): *Algorithm and Programs for Solution of Linear Differential Games* [in Russian], IMM, UNTs, Akad. Nauk SSSR, Sverdlovsk (1984), pp. 5–38.
2. N. D. Botkin, "Optimal universal strategy in a linear differential game," *Differents. Uravn.,* **25**, No. 9, 1475–1480 (1989).
3. N. B. Botkin, V. M. Kein, A. I. Krasov, and V. S. Patsko, *Control of the lateral motion of an aircraft in landing under a wind perturbation* [in Russian], Preprint, VINITI, Moscow (1983).
4. N. D. Botkin, V. M. Kein, and V. S. Patsko, "Model control problem of lateral motion of an aircraft in landing," *Prikl. Mat. Mekh.*, **48**, No. 4, 560–567 (1984).
5. N. D. Botkin, V. M. Kein, and V. S. Patsko, "Solution of the landing problem of an aircraft in the minimax statement," in: *Collection of Scientific Works: Optimization of Control of Aircraft and Their Systems* [in Russian], MAI, Moscow (1988), pp. 8–15.
6. N. D. Botkin, V. M. Kein, and V. S. Patsko, "Application of methods of differential game theory to the control problem of an aircraft in landing," in: *Collection of Scientific Works: Positional Control with a Guaranteed Result* [in Russian], IMM, UrO Akad. Nauk SSSR, Sverdlovsk (1988), pp. 33–44.
7. N. D. Botkin, V. M. Kein, V. S. Patsko, and V. L. Turova, "Aircraft landing control in the presence of windshear," *Probl. Control Inform. Theory*, **18**, No. 4, 223–235 (1989).
8. N. D. Botkin and A. I. Krasov, "Positional control in the model problem of the take-off-run of an aircraft," in: *Collection of Scientific Works: Positional Control with a Guaranteed Result* [in Russian], IMM, UrO Akad. Nauk SSSR, Sverdlovsk (1988), pp. 22–32.
9. N. D. Botkin and V. S. Patsko, "Universal strategy in a differential game with a fixed instant of termination," *Probl. Upr. Teor. Inf.*, **11**, No. 5, 419–432 (1982).
10. N. D. Botkin and V. S. Patsko, "Positional control in a linear differential game," *Izv. Akad. Nauk SSSR., Tekh. Kibern.,* **4**, 78–85 (1983).
11. N. D. Botkin and V. S. Patsko, "Numerical solution of linear differential games," in: *Differential Equations and Applications, I, Rousse, Bulgaria, 1985* [in Russian], Angel Kanchev Tech. Univ, Rousse (1987), pp. 543–546.
12. N. D. Botkin and V. S. Patsko, *Analysis of applications of methods of differential game theory to modeling of wind perturbations* [in Russian], Preprint, VINITI, Moscow (1987).
13. N. D. Botkin, V. S. Patsko, and V. L. Turova, *Elaboration of an algorithm for constructing extremal wind perturbations* [in Russian], Preprint, VINITI, Moscow (1987).
14. N. D. Botkin and V. L. Turova, *Elaboration of applied programs for synthesizing extremal wind perturbations at the stage of landing* [in Russian], Preprint, VINITI, Moscow (1988).

15. B. B. Botkin, V. L. Turova, and A. G. Ivanov, *Recommendation for modeling extremal wind perturbations* [in Russian], Preprint, VINITI, Moscow (1988).
16. N. D. Botkin and M. A. Zarkh, "Error estimate of construction of the positional consumption set in a linear differential game," In: *Collection* (*Materials on Computer Software*): *Algorithms and Programs for Solutions of Linear Differential Games* [in Russian], IMM, UNTs, Akad. Nauk SSSR, Sverdlovsk (1984), pp. 39–80.
17. N. D. Botkin, M. A. Zarkh, V. N. Kein, V. S. Patsko, and V. L. Turova, "Differential games and control problems of an aircraft under wind disturbances," *Izv. Ross. Akad. Nauk, Tekh. Kibern.*, **1**, 68–76 (1993).
18. N. D. Botkin, M. A. Zarkh, and V. S. Patsko, "Numerical solution of linear differential games," in: *Lect. Notes Control Inform. Sci.*, **156**, Springer-Verlag, Berlin–New York (1991), pp. 226–234.
19. N. D. Botkin, S. P. Zhukov, and A. I. Krasnov, "Combined control method of an aircraft in landing," in: *Collection of Scientific Works: Control in Dynamical Systems* [in Russian], IMM, UrO Akad. Nauk SSSR, Sverdlovsk (1990), pp. 18–30.
20. Y. H. Chen and S. Pandey, "Robust control strategy for take-off performance in a windshear," *Optimal Control Appl. Methods*, **10**, No. 1, 65–79 (1989).
21. E. A. Isakova, G. V. Logunova, and V. S. Patsko, "Construction of a stable bridge in a linear differential game with a fixed time of termination," in: *Collection* (*Materials on Computer Software*): *Algorithms and Programs for Solution of Linear Differential Games* [in Russian], IMM, UNTs, Akad. Nauk SSSR, Sverdlovsk (1984), pp. 127–158.
22. M. Ivan, "A ring-vortex downburst model for real-time flight simulation of severe windshears," in: *Flight Simulation Technologies Conf., July 22–24, 1985*, St. Louis (1985), pp. 57–61
23. A. G. Ivanov, "Modeling of motion of an aircraft at the stage of landing," In: *Collection of Scientific Works: Control with a Guaranteed Result* [in Russian], IMM, UrO Ross. Akad. Nauk, Ekaterinburg (1992), pp. 15–26.
24. V. M. Kein, *Optimization of Control Systems with Respect to the Minimax Criterion* [in Russian], Nauka, Moscow (1985).
25. V. M. Kein, V. S. Patsko, and V. L. Turova, "Landing problem of an aircraft under wind shift conditions," In: *Collection of Scientific Works: Control in Dynamical Systems* [in Russian], IMM, UrO Akad. Nauk SSSR, Sverdlovsk (1990), pp. 52–64.
26. V. M. Kein, V. S. Patsko, and L. V. Turova, "Aircraft control under a wind shift," In: *Gagarin Scientific Readings in Cosmic Science and Aviation* [in Russian], Nauka, Moscow (1990), pp. 53–56.
27. N. N. Krasovskii, *Game Problems on the Meeting of Motions* [in Russian], Nauka, Moscow (1970).
28. N. N. Krasovskii, "Differential games, approximate and formal models," *Mat. Sb.*, **107**, No. 4, 541–571 (1978).
29. N. N. Krasovskii and A. I. Subbotin, *Positional Differential Games* [in Russian], Nauka, Moscow (1974).
30. N. N. Krasovskii and A. I. Subbotin, *Game-Theoretical Control Problems*, Springer-Verlag, New York–Berlin (1988).
31. G. Leitmann and S. Pandey, "Aircraft control for flight in an uncertain environment: Take-off in windshear," *J. Optimiz. Theory Appl.*, **70**, No. 1, 25–55 (1991).
32. A. Miele, T. Wang, and W. W. Melvin, "Optimal take-off trajectories in the presence of windshear," *J. Optimiz. Theory Appl.*, **49**, No. 1, 1–45 (1986).
33. A. Miele, T. Wang, C. Y. Tzeng, and W. W. Melvin, "Optimal abort landing trajectories in the presence of windshear," *J. Optimiz. Theory Appl.*, **55**, No. 2, 165–202 (1987).
34. A. Miele, T. Wang, H. Wang, and W. W. Melvin, "Optimal penetration landing trajectories in the presence of windshear," *J. Optimiz. Theory Appl.*, **57**, No. 1, 1–40 (1988).
35. V. S. Patsko, "Special aspects of convex hull constructing in linear differential games of small dimension," in: *Proc. 10th IFAC Workshop "Control Applications of Optimization," December 19–21, 1995, Haifa, Israel*, Pergamon Press (1996), pp. 19–24.

36. V. S. Patsko, "Switching surfaces in linear differential games with a fixed time of termination," *Prikl. Mat. Mekh.*, **68**, No. 4, 653–666 (2004).
37. V. S. Patsko, "Switching surfaces in linear differential games with a fixed time of termination," *Preprint*, IMM, UrO Ross. Akad. Nauk, Ekaterinburg (2004).
38. V. S. Patsko, N. D. Botkin, V. M. Kein, V. L. Turova, and M. A. Zarkh, "Control of an aircraft landing in windshear," *J. Optimiz. Theory Appl.*, **83**, No. 2, 237–267 (1994).
39. B. N. Sokolov and V. L. Turova, *Optimal control of a pendulum under conditions of uncertain disturbances* [in Russian], Preprint No. 336, Inst Prikl. Mat., Akad. Nauk SSSR, Moscow (1988).
40. B. N. Sokolov and V. L. Turova, "Synthesis of optimal control of a pendulum under active disturbances," *Izv. Ross. Akad. Nauk, Mekh. Tverd. Tela*, **23**, No. 5, 14–23 (1988).
41. A. I. Subbotin and A. G. Chentsov, *Optimization of Guarantee in Control Problems* [in Russian], Nauka, Moscow (1981).
42. N. N. Subbotina, "Universal positional strategies in positional differential games," *Differents. Uravn.*, **19**, No. 11, 1890–1896 (1983).
43. V. L. Turova, *Take-off control in a windshear* [in Russian], Preprint, Inst. Math. and Mech., Ekaterinburg (1991).
44. V. L. Turova, *Study of take-off and landing termination problems of an aircraft by using numerical methods of differential game theory* [in Russian], Preprint, VINITI, Moscow (1992).
45. L. V. Turova, "Application of numerical methods of differential game theory to take-off and landing termination problems of an aircraft," *Tr. Inst Mat. Mekh. UrO Ross. Akad. Nauk*, **2**, 188–201 (1992).
46. M. A. Zarkh, *Universal optimal strategy of the second player in a linear differential game* [in Russian], Preprint, VINITI, Moscow (1985).
47. M. A. Zarkh, "Universal optimal strategy of the second player in a linear differential game," *Prikl. Mat. Mekh.*, **54**, No. 3, 395–400 (1990).
48. M. A. Zarkh, "Program package for solution of three-dimensional differential games," in: *Collection of Scientific Works: Analysis and Synthesis of Dynamical Systems under Uncertainty Conditions* [in Russian], MAI, Moscow (1990), pp. 35–41.
49. M. A. Zarkh and V. S. Patsko, *Positional control of the second player in a linear differential game with a fixed instant of termination* [in Russian], Preprint, VINITI, Moscow (1985).
50. M. A. Zarkh and V. S. Patsko, "Construction of control of the second player in a linear differential game on the basis of the repulsion property," In: *Collection of Scientific Works: Control with a Guaranteed Result* [in Russian], IMM, UNTs., Akad. Nauk SSSR, Sverdlovsk (1987), pp. 37–70.
51. S. Zhu and B. Etkin, "Model of wind field in a downburst," *J. Aircraft*, **22**, No. 7, 595–601 (1985).

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