Level Sweeping of the Value Function in Linear Differential Games*

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Abstract

In this chapter one considers a linear antagonistic differential game with fixed terminal time T, geometric constraints on the players' controls, and continuous quasi-convex payoff function φ depending on two components x_i, x_j of the phase vector x. Let $\mathcal{M}_c = \{x : \varphi(x_i, x_j) \leqslant c\}$ be a level set (a Lebesgue set) of the payoff function. One says that the function φ possesses the level sweeping property if for any pair of constants $c_1 < c_2$ the relation $\mathcal{M}_{c_2} = \mathcal{M}_{c_1} + (\mathcal{M}_{c_2} \overset{*}{-} \mathcal{M}_{c_1})$ holds. Here, the symbols $+$ and $\overset{*}{-}$ mean algebraic sum (Minkowski sum) and geometric difference (Minkowski difference). Let $\mathcal{W}_c$ be a level set of the value function $(t, x) \mapsto \mathcal{V}(t, x)$. The main result of this work is the proof of the fact that if the payoff function φ possesses the level sweeping property, then for any $t \in [t_0, T]$ the function $x \mapsto \mathcal{V}(t, x)$ also has the property: $\mathcal{W}_{c_2}(t) = \mathcal{W}_{c_1}(t) + \big(\mathcal{W}_{c_2}(t) \overset{*}{-} \mathcal{W}_{c_1}(t)\big)$. Such an inheritance of the level sweeping property by the value function is specific to the case where the payoff function depends on two components of the phase vector. If it depends on three or more components of the vector x, the statement, generally speaking, is wrong. This is shown by a counterexample.

Key words. Linear differential games, value function, level sets, geometric difference, complete sweeping

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1 Introduction

The central theme for this work is the operation of the geometric difference (Minkowski difference). Its definition and basic properties are given, for example, in [5]. At the early stage of developing the theory of differential games, the geometric difference was applied in [13,14] to solve games with linear dynamics. After that, the concept of the geometric difference was intensively used in the theory of control and differential games (see, for example, [10,3,2,9]).

As usual, the *algebraic sum* (Minkowski sum) of two sets A and B is the set $A + B = \{a + b : a \in A,\ b \in B\}$.

Definition 1.1. The geometric difference of two sets A and B, where $B \neq \varnothing$, is the set $A \overset{*}{-} B = \{x : B + x \subset A\}$. In other words, the geometric difference of the sets A and B is the set of elements such that each of them shifts the set B into the set A.

Let us give some planar examples (Figure 1). The example a) shows the geometric difference of a large square and a small circle. The result is a square with the sides shorter than the original ones by the diameter of the circle. The example b) demonstrates the geometric difference of two circles. The result is also a circle with the radius equal to the difference of the radii of the original circles.

If the set A is convex, then the set $A \overset{*}{-} B$ is convex too. In general the following relation holds:

$$B + (A \overset{*}{-} B) \subset A,$$

that is, the subtrahend set after summation with the geometric difference gives, generally speaking, only a subset of the original set. For instance, in the first

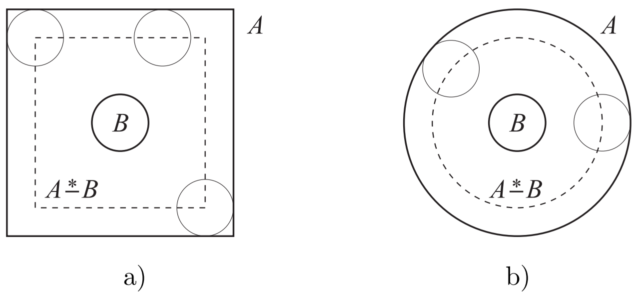


Figure 1: Examples of geometric difference: a) the geometric difference of a square and a circle; b) the geometric difference of two circles. The geometric difference is shown by dashed lines. Thin lines denote some extreme lays of the subtrahend set.

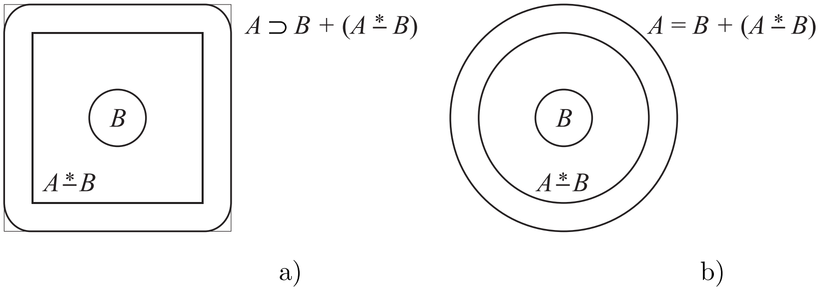


Figure 2: Pictures of the summation of the geometric difference and the subtrahend set for the examples in Figure 1.

of the above examples, after such a summation a square with round corners is obtained (Figure 2a). In the example b), such a summation gives exactly the original circle (Figure 2b).

Definition 1.2. The situation, when the equality

$$B + (A \overset{*}{-} B) = A$$

holds, is called the *complete sweeping* of the set A by the set B.

The notion of "complete sweeping" was originally introduced in [4]. The preceding example a) shows the possibility of absence of the complete sweeping property, whereas example b) shows its possible presence.

As a good illustrative analogy, one can imagine the set A as a room and the set B as a broom. So, the situation of complete sweeping corresponds to a good hostess who sweeps the whole room and does not miss any corner.

Let us give an equivalent definition of the complete sweeping.

Definition 1.3. A set A is completely swept by a set B if $\forall a \in A\, \exists x : 1)\, a \in B + x$ and $2)\, B + x \subset A$.

Let M_c be the level set (the Lebesgue set) of a function f corresponding to a constant c: $M_c = \{x : f(x) \leqslant c\}$.

Definition 1.4. A function f possesses the *level sweeping* property if for any pair of constants $c_1 < c_2$ such that $M_{c_1} \neq \varnothing$, the set M_{c_1} sweeps completely the set M_{c_2}, that is, the relation $M_{c_2} = M_{c_1} + (M_{c_2} \overset{*}{-} M_{c_1})$ holds.

Note that the convexity of a function is neither necessary nor sufficient for presence of the level sweeping property. This is demonstrated by the example shown in Figure 3. Here we consider a function whose graph is a hemisphere cut by two planes such that some smaller level set is a circle and some greater one is a circle with a "roof." It is evident that the smaller level set does not completely sweep the greater one: the corner of the latter cannot be covered.

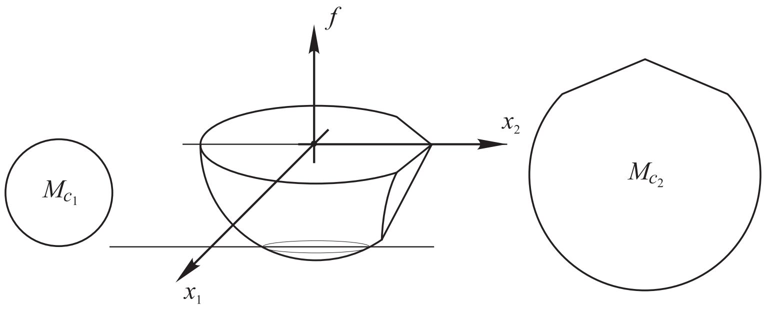


Figure 3: Example of a convex function which does not possess the level sweeping property.

2 Description of the Main Result

Let us consider a linear antagonistic differential game

$$\begin{aligned} &\dot{x} = A(t)x + B(t)u + C(t)v, \quad t \in [t_0, T],\ x \in \mathbb{R}^n,\ u \in P,\ v \in Q, \\ &\varphi\big(x_i(T), x_j(T)\big) \to \min_u \max_v \end{aligned} \tag{1}$$

with fixed terminal time T, convex compact constraints P, Q for controls of the first and second players, and continuous quasi-convex payoff function φ depending on two components x_i, x_j of the phase vector x at the terminal time. (A function is quasi-convex if each of its level sets (Lebesgue sets) is convex.) The first player minimizes the payoff, and the interests of the second one are opposite. It is assumed that every level set $M_c = \big\{(x_i, x_j) : \varphi(x_i, x_j) \leqslant c\big\}$ of the payoff function φ is bounded in the coordinates x_i, x_j.

Using a change of variable $y(t) = X_{i,j}(T, t)x(t)$ ([7, p. 354], [8, pp. 89–91]), which is provided by a matrix combined of two rows of the fundamental Cauchy matrix of system (1), one can pass to the equivalent game

$$\begin{aligned} &\dot{y} = D(t)u + E(t)v, \\ &t \in [t_0, T], \quad y \in \mathbb{R}^2, \quad u \in P, \quad v \in Q, \quad \varphi\big(y_1(T), y_2(T)\big), \\ &D(t) = X_{i,j}(T, t)B(t), \quad E(t) = X_{i,j}(T, t)C(t). \end{aligned} \tag{2}$$

Here, the new phase variable y is two dimensional. The right-hand side of the dynamics does not contain the phase variable. The game interval, the constraints for controls, and the payoff function are the same as in the original game (1) (except that the payoff function now depends on components of the vector y).

Let $(t, y) \mapsto V(t, y)$ be the value function of the differential game (2). The function V is continuous. For any $t \in [t_0, T]$, the function $y \mapsto V(t, y)$ is quasi-convex with compact level sets.

Suppose that the payoff function φ possesses the level sweeping property, that is, for two arbitrary constants $c_1 < c_2$ the corresponding level sets M_{c_1}

and M_{c_2} of the function φ (such that $M_{c_1} \neq \varnothing$) obey the relation

$$M_{c_2} = M_{c_1} + (M_{c_2} \overset{*}{-} M_{c_1}). \tag{3}$$

It turns out that the value function inherits the level sweeping property from the payoff function. Namely, let $W_c(t) = \{y : V(t,y) \leqslant c\}$ be a time section at the instant t of the level set $W_c = \{(t,y) : V(t,y) \leqslant c\}$ of the value function V. In the paper, it is shown that the relation (3) with an additional condition $W_{c_1}(t) \neq \varnothing$, $t \in [t_0, T]$, gives

$$W_{c_2}(t) = W_{c_1}(t) + \bigl(W_{c_2}(t) \overset{*}{-} W_{c_1}(t)\bigr), \qquad t \in [t_0, T]. \tag{4}$$

The main result can be reformulated in the following way.

Theorem 2.1. *If the payoff function of the game* (2) *is such that any of its smaller level sets completely sweeps any larger one, then the time sections of level sets of the value function at any fixed time instant t from the game interval have the same property.*

Since the sections of a level set of the value function in the original and equivalent coordinates are connected by the relation $\mathcal{W}_c(t) = \{x \in \mathbb{R}^n : X_{i,j}(T,t)x \in W_c(t)\}$, $t \in [t_0, T]$, the statement about inheritance of the level sweeping property by the value function from the payoff function is also true for the original game (1). In this form, the fact was formulated in the abstract.

3 Backward Procedure for Constructing Level Sets

To prove the theorem, now a backward procedure will be described, which constructs approximately a level set of the value function in game (2). A level set corresponding to a number c is built as a collection of time sections $\{\boldsymbol{W}_c(t_i)\}$ in a grid of instants $\{t_i\}$. Here, the bold notation $\boldsymbol{W}$ is used instead of W to emphasize that approximate sets are used. Construction is started from a level set M_c of the payoff function taken at the terminal instant T. The set M_c is processed by means of a procedure to the instant $T - \Delta$ giving the section $\boldsymbol{W}_c(T - \Delta)$. Then by means of the same procedure on the basis of the set $\boldsymbol{W}_c(T - \Delta)$, a new set $\boldsymbol{W}_c(T - 2\Delta)$ is computed for the instant $T - 2\Delta$, and so on until the given time $t_* \in [t_0, T)$ (Figure 4).

The procedure for constructing a section $\boldsymbol{W}_c(t_i)$ uses the previous section $\boldsymbol{W}_c(t_{i+1})$ of the level set, the matrices $D(t_i)$ and $E(t_i)$ from the game dynamics (2), and the sets P and Q constraining the players' controls. It is described by the following formula [14,15,9]:

$$\boldsymbol{W}_c(t_i) = \Bigl(\boldsymbol{W}_c(t_{i+1}) + \Delta\bigl(-D(t_i)P\bigr)\Bigr) \overset{*}{-} \Delta E(t_i)Q. \tag{5}$$

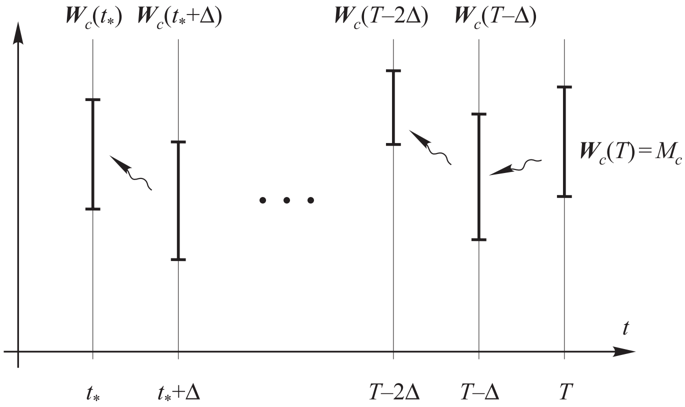


Figure 4: Scheme of the backward procedure of constructing a level set of the value function.

Suppose that $\operatorname{int} W_c(t) \neq \varnothing$ for any $t \in [t_*, T]$. Here, $\operatorname{int} A$ means the interior of a set A. It is known that when decreasing the step size Δ of the discrete scheme, the approximately built section $\boldsymbol{W}_c(t_*)$ of a level set converges to the ideal one $W_c(t_*)$ in the Hausdorff metric [12,1,11].

So, to prove the inheritance of the level sweeping property by the value function it is necessary to prove that the property of complete sweeping is conserved after operations of algebraic sum and geometric difference and after passing to the limit when decreasing the step size Δ.

4 Additional Properties of the Geometric Difference

The following statement concerns the conservation of the complete sweeping property after the operations of algebraic sum and geometric difference.

Lemma 4.1. *Let convex compact sets A, B, and C in the plane be such that the set A is completely swept by the set B, that is, $A = B + (A \overset{*}{-} B)$. Then*
1) $(A + C) = (B + C) + \big((A + C) \overset{*}{-} (B + C)\big)$;
2) *if $B \overset{*}{-} C \neq \varnothing$, then* $(A \overset{*}{-} C) = (B \overset{*}{-} C) + \big((A \overset{*}{-} C) \overset{*}{-} (B \overset{*}{-} C)\big)$.

Proof. The first fact is proved directly with the help of equivalent Definition 1.3 of the complete sweeping. So, let us show that for any $a' \in A + C$ there is an element $x \in \mathbb{R}^2$ such that $a' \in (B + C) + x$ and $(B + C) + x \subset (A + C)$.

Fix $a' \in A + C$. Then one can find $a \in A$ and $c \in C$ such that $a' = a + c$. According to the complete sweeping of the set A by the set B, there is an element $x \in \mathbb{R}^2$ such that $a \in B + x$ and $B + x \subset A$. Prove that this element x is also acceptable for establishing the complete sweeping of the set $A + C$ by the set $B + C$.

Since $a \in B + x$, it follows that $a + c = a' \in B + x + c \subset (B + C) + x$.

Because $B + x \subset A$, then $(B + C) + x \subset (A + C)$.

So, the conservation of the complete sweeping after the algebraic sum is proved. Note that this proof does not demand any compactness, or convexity, or dimension restriction of the sets A, B, and C. Therefore, statement 1) of Lemma 4.1 also holds under more general conditions.

Let us proceed to statement 2) of Lemma 4.1. We use the support functions of the sets under consideration. Recall that every convex compact set A produces a finite positively homogeneous convex function by the formula $\rho_A(l) = \max\{l'a : a \in A\}$. This function is called the *support function* of the set A. And vice versa, for any finite positively homogeneous convex function ρ, a convex compact set can be found such that ρ is its support function [16].

Let us establish a correspondence between set operations and operations over support functions. Let $A \leftrightarrow \rho_A$, $B \leftrightarrow \rho_B$. Then $\rho_{A+B} = \rho_A + \rho_B$. It is also known that if $A \overset{*}{-} B \neq \varnothing$, then $\rho_{A \overset{*}{-} B} = \operatorname{conv}\{\rho_A - \rho_B\}$ [2,9]. When $A \overset{*}{-} B = \varnothing$, it is supposed that $\rho_{A \overset{*}{-} B} \equiv -\infty$.

Let the set A be completely swept by the set B, that is, $A = B + (A \overset{*}{-} B)$. Then $\rho_A = \rho_B + \operatorname{conv}\{\rho_A - \rho_B\}$, or $\rho_A - \rho_B = \operatorname{conv}\{\rho_A - \rho_B\}$. Hence, if the set A is completely swept by the set B, then the difference of their support functions is convex.

Using the language of support functions, the statement about conservation of the complete sweeping property after the geometric difference can be formulated as follows.

2*) Let some convex compact sets A, B, and C be such that the difference $\rho_A - \rho_B$ is convex and the function $\operatorname{conv}\{\rho_B - \rho_C\}$ has finite value everywhere in $\mathbb{R}^2$. Then the difference $\operatorname{conv}\{\rho_A - \rho_C\} - \operatorname{conv}\{\rho_B - \rho_C\}$ is also convex.

Assume $f = \rho_A - \rho_C$, $g = \rho_B - \rho_C$.

The function $f - g = (\rho_A - \rho_C) - (\rho_B - \rho_C) = \rho_A - \rho_B$ is convex. Convexity of the function $\operatorname{conv} f - \operatorname{conv} g = \operatorname{conv}\{\rho_A - \rho_C\} - \operatorname{conv}\{\rho_B - \rho_C\}$ is shown in the next lemma. □

Lemma 4.2. *Let functions f and $g : \mathbb{R}^2 \to \mathbb{R}$ be positively homogeneous, continuous, the difference $f - g$ be convex, and the function* $\operatorname{conv} g$ *have finite value everywhere in $\mathbb{R}^2$. Then the difference* $\operatorname{conv} f - \operatorname{conv} g$ *is a convex function.*

Before the proof of Lemma 4.2, let us formulate some auxiliary propositions. They are quite simple, so no proofs are given.

Let us denote the boundary of a set D by ∂D. Restriction of f to a set D will be written as $f\big|_D$. By $\operatorname{conv}\big|_D f$ we mean the convex hull of the function f computed in a convex set D.

1° Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function. Also let $D \subset \mathbb{R}^n$ be a closed convex set and let the function $\tilde{f}$ be convex in the set D. Let us suppose that $\tilde{f}(x) = f(x)$

when $x \in \partial D$ and $\tilde{f}(x) \geqslant f(x)$ when $x \in \operatorname{int} D$. Then the function

$$g(x) = \begin{cases} \tilde{f}(x), & x \in D, \\ f(x), & x \notin D \end{cases}$$

is convex in $\mathbb{R}^n$.

2° Let $f : \mathbb{R}^n \to \mathbb{R}$ and $D \subset \mathbb{R}^n$ be a closed convex set. Let us suppose that $(\operatorname{conv} f)(x) = f(x)$ when $x \in \partial D$. Then $\operatorname{conv}\big|_D f = (\operatorname{conv} f)\big|_D$.

3° Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous, positively homogeneous function. Then for any vector $l_* \neq 0$ a vector $p \in \{x : l'_* x \geqslant 0\}$ exists such that $f(p) = (\operatorname{conv} f)(p)$.

4° Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a continuous, positively homogeneous function, and let C be a closed cone of angle not greater than π. Let us suppose that $f(x) = (\operatorname{conv} f)(x)$ if $x \in \partial C$ and $f(x) \neq (\operatorname{conv} f)(x)$ if $x \in \operatorname{int} C$. Then the function $\operatorname{conv} f$ is linear in the cone C.

Now, Lemma 4.2 will be proved.

Proof. 1) Let us denote $\tilde{g} = \operatorname{conv} g$, $S = \{x \in \mathbb{R}^2 : \tilde{g}(x) = g(x)\}$. By virtue of the continuity of the functions $\tilde{g}$ and g, the set S is closed. Thus, the set $\mathbb{R}^2 \backslash S$ can be presented as at most a countable join of non-overlapping open cones C_i^0, $i = \overline{1, m}$, $m \leqslant \infty$. Following proposition 3°, each of these cones is of angle not greater than π. Let C_i be the closure of the cone C_i^0.

Using proposition 2°, one can establish that for any i, the equality $\operatorname{conv}\big|_{C_i} g = (\operatorname{conv} g)\big|_{C_i}$ holds.

2) The process of constructing the convex hull of the function g can be considered as a stepwise one: $g = g_0 \rightsquigarrow g_1 \rightsquigarrow g_2 \rightsquigarrow \ldots$ Here, each next function g_i is obtained from the previous one g_{i-1} by changing the latter in the cone C_i by a linear function l_i. One has $l_i(x) = g_{i-1}(x)$ when $x \in \partial C_i$ and $l_i(x) < g_{i-1}(x)$ when $x \in \operatorname{int} C_i$. Also according to proposition 4°, $l_i = (\operatorname{conv} g)\big|_{C_i}$.

Simultaneously, the function f is also corrected: $f = f_0 \rightsquigarrow f_1 \rightsquigarrow f_2 \rightsquigarrow \ldots$ such that $f_i\big|_{C_i} = \operatorname{conv}\big|_{C_i} f_{i-1}$, $f_i\big|_{\mathbb{R}^2 \backslash C_i} = f_{i-1}\big|_{\mathbb{R}^2 \backslash C_i}$. That is, f_i is obtained from f_{i-1} by convexification of the latter in the cone C_i.

3) Let $h_i = f_i - g_i$, $i \geqslant 0$. We will prove by induction on i that for any i the function h_i is convex.

When $i = 0$, the function $h_0 = f_0 - g_0 = f - g$ is convex by the condition of the lemma.

Suppose that for any $0 \leqslant i - 1 < m$, the function h_{i-1} is convex. We will show that in this case the function h_i is also convex.

When $x \in \mathbb{R}^2 \backslash C_i$, one has $g_i(x) = g_{i-1}(x)$ and $f_i(x) = f_{i-1}(x)$. Therefore, $h_i = h_{i-1}$ in $\mathbb{R}^2 \backslash C_i$.

We have $g_i(x) \leqslant g_{i-1}(x)$ when $x \in C_i$. Thus, in the cone C_i the relation $f_{i-1} - g_i \geqslant f_{i-1} - g_{i-1} = h_{i-1}$ holds, and, therefore, $f_{i-1} \geqslant g_i + h_{i-1}$. Because g_i is linear in C_i, then the sum $g_i + h_{i-1}$ is convex in C_i. Consequently, it follows that in C_i the relation $f_i = \operatorname{conv}\big|_{C_i} f_{i-1} \geqslant g_i + h_{i-1}$ holds, that is, $h_i = f_i - g_i \geqslant h_{i-1}$.

Since in the cone C_i the function f_i is convex and g_i is linear, the function $h_i = f_i - g_i$ is convex in C_i.

Applying proposition 1°, one obtains that the function h_i is convex in $\mathbb{R}^2$.

4) The sequence of the continuous functions g_i is nonincreasing. With that $\lim g_i = \operatorname{conv} g$. The sequence of the continuous functions f_i is nonincreasing and is bounded from below by the function $\operatorname{conv} f$. Thus, this sequence has a pointwise limit $\tilde{f}$. The sequence of convex functions h_i converges pointwise to a convex function $\tilde{h} = \tilde{f} - \operatorname{conv} g$. Hence, the function $\tilde{f} = \tilde{h} + \operatorname{conv} g$ is convex.

Let us prove that $\tilde{f} = \operatorname{conv} f$. One has that $\tilde{f}(x) = f(x) \geqslant (\operatorname{conv} f)(x)$ when $x \in S$. For any $x \in \mathbb{R}^2 \backslash S$ an index $i \geqslant 1$ exists such that $x \in C_i$, and, therefore,

$$\tilde{f}(x) = f_i(x) = \left(\operatorname{conv}\big|_{C_i} f_{i-1}\right)(x) = \left(\operatorname{conv}\big|_{C_i} f\right)(x) \geqslant (\operatorname{conv} f)(x).$$

Hence, $\tilde{f} \geqslant \operatorname{conv} f$. Because $f \geqslant \tilde{f}$ and the function $\tilde{f}$ is convex, then $\tilde{f} = \operatorname{conv} f$.

By this, it is shown that the difference $\operatorname{conv} f - \operatorname{conv} g$ is convex in $\mathbb{R}^2$. □

5 Counterexamples to Generalizations of Lemma 4.2

Note that Lemma 4.2 holds only for positive homogeneous functions of two variables. Generally speaking, the lemma does not hold if the function does not possess positive homogeneity or the dimension of its argument is higher than two.

Let us show this by some counterexamples. At first, an example of convex compact three-dimensional sets A, B, and C will be given such that the set B completely sweeps the set A, but the difference $B \overset{*}{-} C$ does not completely sweep the set $A \overset{*}{-} C$. Let us take the set A as a hemisphere cut by two planes (Figure 5). The set B is homothetic to the set A with coefficient of homothety less than 1. The set C is taken as an interval, where the length is less than the horizontal side of the cut part of the set A, but larger than the cut part of the set B.

Since the set C is an interval, the geometric difference $B \overset{*}{-} C$ ($A \overset{*}{-} C$) is the intersection of two copies of the set B (correspondingly, A) shifted by the length of the interval C. According to this, the difference $B \overset{*}{-} C$ looks like a cap: the cut part disappeared. At the same time, the difference $A \overset{*}{-} C$ keeps the cut part. The sections of the flat sides of the geometric differences are shown at the right in Figure 5. It is evident that the sharp point of the "roof" of the set $A \overset{*}{-} C$ cannot be covered by the circle $B \overset{*}{-} C$. Therefore, there is no complete sweeping between the sets $A \overset{*}{-} C$ and $B \overset{*}{-} C$.

Thus, a counterexample for a possible generalization of statement 2) of Lemma 4.1 is constructed for the case when the sets A, B, C are of dimension

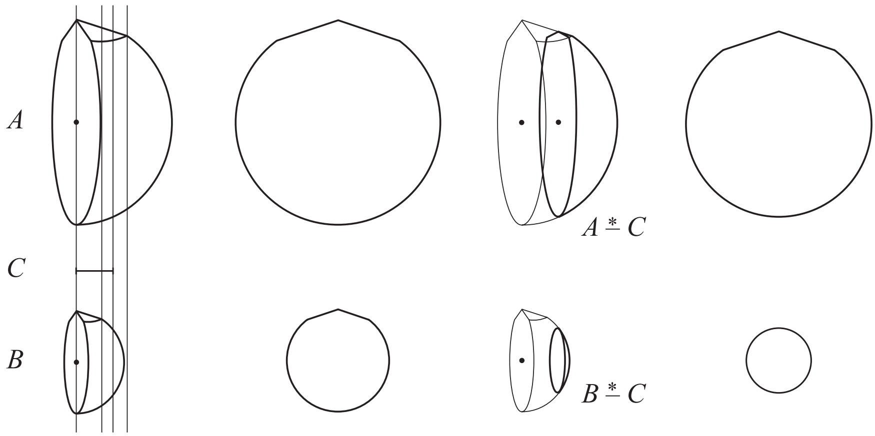


Figure 5: Counterexample for conservation of the complete sweeping after the operation of geometric difference of three-dimensional sets.

higher than two. Support functions of the sets considered give a counterexample for a generalization of statement 2*) and, therefore, for Lemma 4.2 in the case when the positively homogeneous functions have their arguments of dimension three or higher.

Violation of Lemma 4.2 in the case of functions of the general kind (not positively homogeneous) is demonstrated by the following example.

Let the functions f and g be piecewise linear. The graph of the function f can be obtained from a quadrahedral pyramid by cutting it by two planes parallel to the diagonal of the base (Figure 6a). Something looking like a "chisel" appears.

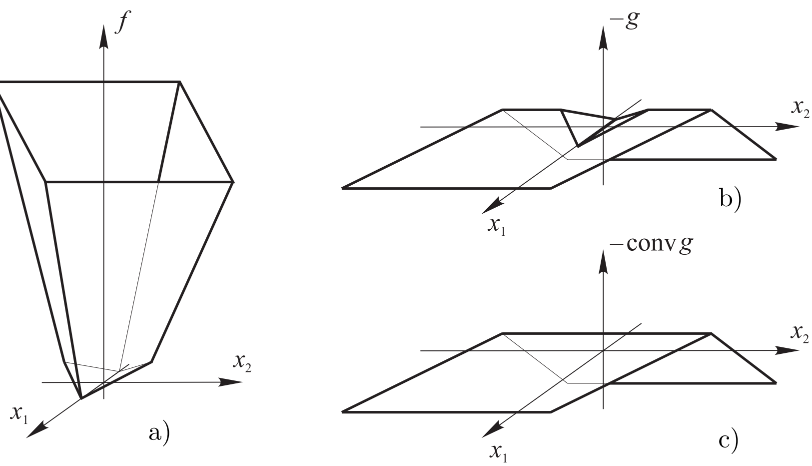
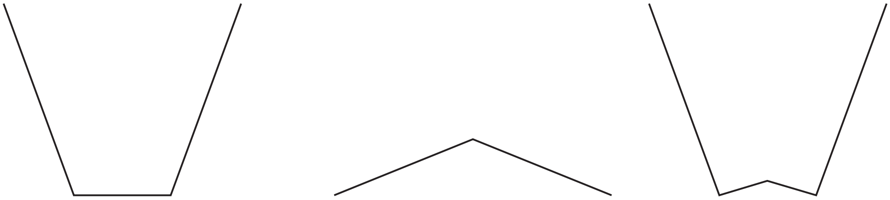


Figure 6: Graphs of the functions f (a), $-g$ (b), and $-\mathrm{conv}\, g$ (c).



a) b) c)

Figure 7: Sections of the graphs of $\operatorname{conv} f = f$ (a), $-\operatorname{conv} g$ (b), and $\operatorname{conv} f - \operatorname{conv} g$ (c).

The graph of the function $-g$ (it is more demonstrative to imagine the function $-g$) looks like a "roof" having a cavity of the same form as the bottom of the graph of f (Figure 6b). The origin is placed at the middle of the bottom of the graph of f and at the middle of the cavity of $-g$. Then the graph of $f - g = f + (-g)$ looks like the graph of f. The slope of the bottom outshoot becomes "sharper" and the slope of the side faces becomes, conversely, "flatter" in comparison with the graph of f. The original slopes can be chosen such that the graph of $f - g$ will be convex. (Namely, it is necessary to take the side faces of f quite "sharp" and the faces of g and the bottom outshoot of f quite "flat.")

Let us consider the graph of the function $\operatorname{conv} f - \operatorname{conv} g = f + (-\operatorname{conv} g)$. The convex hull $\operatorname{conv} f$ coincides with f itself because the function f is convex. The graph of $-\operatorname{conv} g$ (or of the concave hull of $-g$) looks like a "roof" without any cavities (Figure 6c). Let us take the sections of the graphs made by a vertical plane containing the bottom line of "chisel" f. Since the section of the function $\operatorname{conv} f - \operatorname{conv} g$ is non-convex (Figure 7), the function $\operatorname{conv} f - \operatorname{conv} g$ itself is non-convex.

6 Conservation of Level Sweeping to the Limit

Fix an arbitrary instant $t_* \in [t_0, T)$ and choose a sequence $\{\vartheta_k\}$ of subdivisions of the time interval $[t_*, T]$: $\vartheta_k = \{t_* = t_*^{(k)} < \cdots < t_{N_k}^{(k)} = T\}$. With $k \to 0$ diameter Δ_k of subdivision ϑ_k goes to 0. Denote by $\boldsymbol{W}_{c_1}^{(k)}(t_*)$ and $\boldsymbol{W}_{c_2}^{(k)}(t_*)$ the results of applying the backward procedure (5) on the subdivision ϑ_k with starting sets $\boldsymbol{W}_{c_1}(T) = M_{c_1}$ and $\boldsymbol{W}_2(T) = M_{c_2}$.

Because the starting sets $\boldsymbol{W}_{c_1}(T)$ and $\boldsymbol{W}_{c_2}(T)$ have the complete sweeping, then according to the results on conservation of the complete sweeping after algebraic sum and geometric difference from Section 4, each pair of sets $\boldsymbol{W}_{c_1}^{(k)}(t_i)$ and $\boldsymbol{W}_{c_2}^{(k)}(t_i)$ has the complete sweeping. Consequently, for any k the set $\boldsymbol{W}_{c_1}^{(k)}(t_*)$ completely sweeps the set $\boldsymbol{W}_{c_2}^{(k)}(t_*)$.

1) Under the assumption that for any $t \in [t_*, T]$ the section $W_{c_1}(t)$ of ideal level set W_{c_1} of the value function has a non-empty interior (that is, $\operatorname{int} W_{c_1}(t) \neq \varnothing$),

one has the following convergence $\boldsymbol{W}_{c_1}^{(k)}(t_*) \to W_{c_1}(t_*)$ and $\boldsymbol{W}_{c_2}^{(k)}(t_*) \to W_{c_2}(t_*)$ in the Hausdorff metric with $k \to \infty$.

Therefore, to prove the complete sweeping of the set $W_{c_2}(t_*)$ by the set $W_{c_1}(t_*)$ under the additional condition $\operatorname{int} W_{c_1}(t) \neq \varnothing$, $t \in [t_*, T]$, it is necessary to justify the following simple fact. Let two sequences $\{A_k\}$ and $\{B_k\}$ of compact sets converge in the Hausdorff metric to compact sets A and B respectively. Suppose that for any k the set B_k completely sweeps the set A_k. Then the limit sets have the same property: the set B completely sweeps the set A.

Let us show that for the sets A and B, the properties, which stipulate the complete sweeping of the first set by the second one, hold: 1) $\forall a \in A \; \exists x : a \in B + x$ and 2) $B + x \subset A$ (see Definition 1.3).

Fix an arbitrary element $a \in A$. Due to the convergence $A_k \to A$, one can choose a sequence $\{a_k\}$, $a_k \in A_k$, such that $a_k \to a$. Since the set A_k is completely swept by the set B_k, it implies $\forall k \; \exists x_k : a_k \in B_k + x_k$ and $B_k + x_k \subset A_k$.

Consider the sequence $\{x_k\}$. It is bounded. Therefore, a converging subsequence can be extracted from it. Without loss of generality, let us suppose that the sequence $\{x_k\}$ itself converges to an element x. This limit is just the desired element, which figures in the properties giving the complete sweeping. Let us show this fact.

The first property: $a \in B + x$. We have that $\forall k \; a_k \in B_k + x_k$. Choose $b_k \in B_k : a_k = b_k + x_k$. Since $a_k \to a$ and $x_k \to x$, it follows $b_k \to b = a - x$. Taking into account the convergence $B_k \to B$, one can obtain that $b \in B$. Therefore, there is an element $b \in B$ such that $a = b + x$. Consequently, $a \in B + x$.

The second property: $B + x \subset A$. Let us take an arbitrary element $b \in B$. Due to the convergence $B_k \to B$, one can take a sequence $\{b_k\}$, $b_k \in B_k$, such that $b_k \to b$. Since $B_k + x_k \subset A_k$, it implies $b_k + x_k \in A_k$. Therefore, $\forall k \; \exists \, a_k \in A_k : b_k + x_k = a_k$. Because $b_k \to b$ and $x_k \to x$, then a_k tends to an element $\bar{a} = b + x$. Taking into account the convergence $A_k \to A$, one can obtain that $\bar{a} \in A$. This shows that $\forall \, b \in B \;\; b + x \in A$. Consequently, $B + x \subset A$.

Hence, the set B completely sweeps the set A.

2) Now let $W_{c_1}(t_*) \neq \varnothing$, but $\operatorname{int} W_{c_1}(\bar{t}) = \varnothing$ at an instant $\bar{t} \in [t_*, T]$. From the continuity of the value function, it follows that $\operatorname{int} W_c(\bar{t}) \neq \varnothing$ for $c > c_1$. Then also $\operatorname{int} W_c(t) \neq \varnothing$ for $c > c_1$ when $t \in [t_*, T]$. According to the fact proved above, the set $W_c(t_*)$ completely sweeps the set $W_{c_2}(t_*)$ for $c \in (c_1, c_2)$. It follows from this that the set $W_{c_1}(t_*)$ completely sweeps the set $W_{c_2}(t_*)$.

7 Is It Possible to Weaken the Dimension Assumption?

Theorem 2.1 is formulated for the case when the payoff function φ depends on two components of the phase vector at the terminal instant T. Let us show that, generally speaking, the theorem does not hold if the payoff function is defined by three or more components of the phase vector.

Let us consider a differential game

$$\begin{aligned} &\dot{x} = u + v, \quad t \in [t_0, T],\ x \in \mathbb{R}^3,\ u \in \{0\},\ v \in Q, \\ &\varphi\big(x(T)\big) = \min\big\{c : x(T) \in cM\big\} \end{aligned} \tag{6}$$

with fixed terminal time T, a fictitious first player (actually, the first player is absent) and the payoff function, which is the Minkowski function of a compact convex set M. The set M is taken as the set A shown in Figure 5. The payoff function depends on full a three-dimensional phase vector and, evidently, possesses the level sweeping property. As the set Q constraining the control of the second player, let us take the interval shown in Figure 5 and denoted there by C.

Because the right-hand side of the game dynamics does not depend on time and does not contain the phase variable, then for any t and any c the section $W_c(t)$ of the level set of the value function is defined by the formula $W_c(t) = W_c(T) \overset{*}{-} (T-t)Q$. Let $t = T-1$. Take $c_2 = 1$ and $c_1 < 1$ such that the set $M_{c_1} = c_1 M$ coincides with the set B drawn in Figure 5. Then $W_{c_1}(t) = M_{c_1} \overset{*}{-} Q = B \overset{*}{-} C$ and $W_{c_2}(t) = M_{c_2} \overset{*}{-} Q = A \overset{*}{-} C$. As described in Section 5 in the text relating to Figure 5, the set $A \overset{*}{-} C$ is not completely swept by the set $B \overset{*}{-} C$. Therefore, the set $W_{c_2}(t)$ is not completely swept by the set $W_{c_1}(t)$.

Thus, the condition of Theorem 2.1 connected to the number of arguments of the payoff function is essential.

8 Conclusion

In this chapter, a linear antagonistic differential game with fixed terminal time, geometric constraints on the players' controls, and continuous quasi-convex terminal payoff function depending on two components of the phase vector is considered. A level sweeping property of a quasi-convex function is defined. This property consists of the condition that any non-empty smaller level set completely sweeps any larger one. The term "complete sweeping" is based on the concept of geometric difference (Minkowski difference) and is known in convex analysis and in differential game theory. It is proved that, in the game class considered, the level sweeping property is inherited by the value function. That is, if the payoff function possesses the level sweeping property, then the same property is true for the constriction of the value function to any time instant from the game interval. It is shown (by a counterexample) that this holds only when the payoff function depends on at most two components of the phase vector.

The level sweeping property of the value function can be useful, for example, when analyzing singular surfaces appearing in linear differential games with fixed terminal time. Namely, under the presence of this property, the structure of singular surfaces has some patterns absent in the general situation. In this case, numerical algorithms for constructing and classifying singular surfaces become essentially easier.

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