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On Time-Optimal Problems with Lifeline

Nataly V. Munts[1] · Sergey S. Kumkov[1]

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Abstract
This paper discusses time-optimal games with lifeline and corresponding boundary value problems for Hamilton–Jacobi equation as well. Existence of the value function for the time-optimal games with lifeline is proved. Existence of a minimax solution and its coincidence with the value function are shown.

Keywords Time-optimal differential games · Games with lifeline · Value function · Hamilton–Jacobi equations · Minimax solutions

1 Introduction

This paper investigates time-optimal differential games with lifeline, the question of existence of the value function of a game of such a kind, and the connection between the value function and a minimax solution of the corresponding Hamilton–Jacobi equation (HJE). In games of this type, the first player tends to lead the system to a prescribed closed target set while keeping the trajectory inside some open set, where the game takes place. The second player hinders this, because it wins as soon as either the trajectory of the system leaves this open set not touching the target one, or it succeeds in infinitely keeping the system inside this open set.

Apparently, the first, who formulated a problem with lifeline, was Isaacs in his book [11]. In his definitions, the *lifeline* is a set, after reaching which the second player wins unconditionally. Significant contribution into researching games with lifeline was made by Petrosyan (see e.g., [14]). However, authors are not familiar with works, which would exhaustively consider games of this sort: Petrosyan generally researched problems with simple motions dynamics. In books [12,13] of Krasovskii and Subbotin, such games are analyzed as problems with state constraints: The first player is not supposed to lead the system outside a prescribed set. Also, problems with state constraints have been studied by many authors (see, e.g., [3–5,10,15]).

Problems very close to games with lifeline have been studied by French authors Cardaliaguet, Quincampoix, Saint-Pierre [6–9]. For controlled systems on the basis of the

✉ Nataly V. Munts
natalymunts@gmail.com

Sergey S. Kumkov
sskumk@gmail.com

1 Krasovskii Institute of Mathematics and Mechanics, UrB RAS, Yekaterinburg, Russia

set-valued analysis, the theory of differential inclusions, and the theory of viability, they analyzed the sets where the controller is able to keep the system forever (viability kernels). Passing to games, the authors considered a situation with two target sets for the first and second players, respectively, to which the players try to guide the system avoiding the target of the opposite player. Another variant considered in these works is games with state constraints for the first player. In these situations, the main objectives are to study victory domains of the players, that is, the sets wherefrom the corresponding player can reach its target without hitting the target of the opposite player (or state constraints). Also, in their works, the upper value function of such games (the guaranteed result of the first player) is characterized in the terms of viability as a function, which epigraph is a viability set of the first player. Grid-geometric algorithms have been suggested for approximation of viability kernels and, therefore, for approximation of the upper value. However, we have not found papers of these authors, where existence of the value function is proved for games of this type and/or its coincidence with the generalized solution of the corresponding boundary value problem of HJE is proved (although such a connection is mentioned).

The main boost that stimulated the authors for study of time-optimal games with lifeline is the investigation of questions connected to numerical methods for solving classic time-optimal games. In particular, in works [1,2], Italian mathematicians Bardi and Falcone suggested a theoretic numerical method for constructing the value function of a time-optimal game as a generalized (viscosity) solution of the corresponding boundary value problem for HJE. The suggested procedure is of grid character, and its proof has been made in assumption that the grid is infinite and covers the entire game space. But a practical computer realization, apparently, deals with a finite grid, which covers only a bounded part of the game space. So, the problem arises of what boundary condition to set on the outer boundary of the domain covered by the grid. Bardi and Falcone suggested to set these conditions to plus infinity. So, they actually declared that the second player wins when reaching the outer boundary of this domain. Therefore, the practical realization of the procedure solves a game with lifeline. That is why the authors decided to fill this gap connected to the problems with lifeline in a very general formulation.

This paper discusses various aspects of the time-optimal games with lifeline: existence of the value function, existence of a generalized solution of the corresponding boundary value problem for PDE of Hamilton–Jacobi type, and the coincidence of the value function and the generalized solution. In comparison with the French authors, the main tool used in the paper is the concept of stability (which, however, is quite close to the concept of viability). During consideration of HJE, the authors involve the concept of minimax generalized solution and results by Subbotin [16].

The structure of the paper is the following. In Sect. 2, we state the time-optimal problem with lifeline and define the result of the game and necessary conditions. In Sect. 3, we introduce the value function of this problem and prove that the value function exists. In Sect. 4, we give basic definitions of minimax solutions and prove that the minimax solution of the corresponding boundary value problem for HJE exists under some conditions. In Sect. 5, we show that the value function of a time-optimal game with lifeline coincides with the minimax solution (under the same conditions). Finally, in Sect. 6, we give some comments on classic time-optimal games.

2 Problem Formulation

Let us consider a system, whose dynamics is

$$\dot{x} = f(x, a, b), \qquad t \geq 0,\ a \in A,\ b \in B, \tag{1}$$

where $x \in \mathbb{R}^n$ is the phase vector of the system; a and b are the controls of the first and second player. We are given a compact set $\mathcal{T}$ and an open set $\mathcal{W}$ such that $\mathcal{T} \subset \mathcal{W}$ and the boundary $\partial\mathcal{W}$ is bounded. Denote $\mathcal{G} := \mathcal{W} \setminus \mathcal{T}$ and $\mathcal{F} := \mathbb{R}^n \setminus \mathcal{W}$. (Here and below, the symbols ":=" and "=:" mean "is equal by definition" where the variable to be defined is at the side of the colon.) The set $\mathcal{T}$ is the target set; in the set $\mathcal{G}$, the game takes place; the set $\partial\mathcal{F} = \partial\mathcal{W}$ is the lifeline where the second player wins unconditionally (see Fig. 1). In other words, the objective of the first player is to guide the system to the set $\mathcal{T}$ as soon as possible keeping the trajectory outside the set $\mathcal{F}$; the objective of the second player is to guide the system to the set $\mathcal{F}$, or if it is impossible, to keep the trajectory inside the set $\mathcal{G}$ forever, or if the latter is impossible too, to postpone the reaching the set $\mathcal{T}$ as long as he can.

The players' aims of this kind can be formalized in the following way. Let $x(\cdot; x_0)$ be the trajectory emanated from the initial point $x(0) = x_0$. We consider two instants

$$t_* = t_*\big(x(\cdot; x_0)\big) = \min\big\{t \geq 0 : x(t; x_0) \in \mathcal{T}\big\},$$
$$t^* = t^*\big(x(\cdot; x_0)\big) = \min\big\{t \geq 0 : x(t; x_0) \in \mathcal{F}\big\},$$

which are the instants when the trajectory $x(\cdot; x_0)$ for the first time hits the sets $\mathcal{T}$ and $\mathcal{F}$, respectively. If the trajectory does not arrive at the set $\mathcal{T}$ $(\mathcal{F})$, then the value t_* (t^*) is equal to $+\infty$. We define the result of the game as

$$\tau\big(x(\cdot; x_0)\big) = \begin{cases} +\infty, & \text{if } t_* = +\infty \text{ or } t^* < t_*, \\ t_*, & \text{otherwise.} \end{cases} \tag{2}$$

We assume that the following conditions are fulfilled:

C.1 The function $f : \mathbb{R}^n \times A \times B \mapsto \mathbb{R}^n$ is continuous and is Lipschitz continuous on the variable x: for all $x^{(1)}, x^{(2)} \in \mathbb{R}^n$, $a \in A$, $b \in B$

$$\big\| f(x^{(1)}, a, b) - f(x^{(2)}, a, b)\big\| \leq \lambda \|x^{(1)} - x^{(2)}\|; \tag{3}$$

moreover, it satisfies Isaacs' condition:

$$\min_{a\in A} \max_{b\in B} \big\langle p,\ f(x, a, b)\big\rangle = \max_{b\in B} \min_{a\in A} \big\langle p,\ f(x, a, b)\big\rangle =: \mathcal{H}(x, p) \quad \forall\, p \in \mathbb{R}^n. \tag{4}$$

Here and below, the symbol $\langle\cdot, \cdot\rangle$ stands for scalar product.

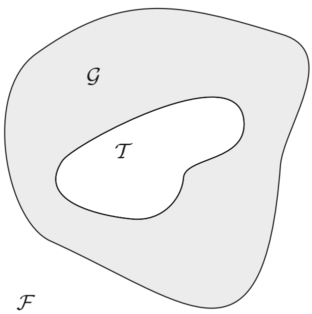

Fig. 1 Sets $\mathcal{T}$, $\mathcal{F}$, and $\mathcal{G}$

C.2 A and B are compact metric spaces.
C.3 The boundary $\partial\mathcal{G}$ (that is the boundaries $\partial\mathcal{T}$ and $\partial\mathcal{F}$) is smooth.
C.4 The boundary $\partial\mathcal{T}$ of the set $\mathcal{T}$ and the function f obey the following condition:

$$\min_{a\in A}\max_{b\in B}\big\langle n_{\mathcal{T}}(x),\, f(x,a,b)\big\rangle < 0$$

for every point $x \in \partial\mathcal{T}$. Here and below, the symbol $n_{\mathcal{A}}(x)$ denotes an outward normal vector to the set $\mathcal{A}$ at the point x, which has a unit length. In terms of Isaacs' book [11], this condition means that the boundary $\partial\mathcal{T}$ is an *admissible zone* for the first player: If the system is at the boundary of the terminal set $\mathcal{T}$, then the first player can guarantee leading the trajectory of the system inside the set.
C.5 The boundary $\partial\mathcal{F}$ of the set $\mathcal{F}$ and the function f obey the following condition:

$$\min_{a\in A}\max_{b\in B}\big\langle -n_{\mathcal{F}}(x),\, f(x,a,b)\big\rangle > 0$$

for every point $x \in \partial\mathcal{F}$. This relation means that if the system is situated at any point of the boundary of the set $\mathcal{F}$, then the second player can direct the system motion into the set $\mathcal{F}$.

3 Value Function

Let us introduce the value function of a time-optimal problem with lifeline in the same way as it is described in [12,13].

We denote $\mathcal{T}_\varepsilon = \mathcal{T}$, $\mathcal{F}_\varepsilon = \mathcal{F} + B_\varepsilon$, $\mathcal{G}_\varepsilon = R^n \setminus (\mathcal{F}_\varepsilon \cup \mathcal{T}_\varepsilon)$. Here and below, B_ε is a closed ball with the radius ε and the center at the origin. The sign "+" for set operands means the algebraic sum (the Minkowski sum). The value of ε is taken such that the set $\mathcal{G}_\varepsilon$ is not empty. For a trajectory $x(\cdot)$ of the system, we define a functional

$$\tau_\varepsilon\big(x(\cdot)\big) := \min\big\{t \in \mathbb{R}^+ : x(t) \in \mathcal{T}_\varepsilon,\ \forall\,\vartheta \in [0,t)\ x(\vartheta) \in \mathcal{G}_\varepsilon\big\}. \tag{5}$$

Again, if the trajectory never reaches the set $\mathcal{T}_\varepsilon$ or reaches $\mathcal{F}_\varepsilon$ earlier than $\mathcal{T}_\varepsilon$, then $\tau_\varepsilon\big(x(\cdot)\big) = +\infty$.

Take a feedback strategy of the first player $\mathcal{A} : \mathbb{R}^n \to A$ (an arbitrary function mapping $\mathbb{R}^n$ to A), an initial point $x_0 \in \mathbb{R}^n$, and a time partition

$$\Delta = \{0 = t_1 < t_2 < t_3 < \ldots\}.$$

Denote by $\mathbb{X}(\bar{x}, \mathcal{A}, \Delta)$ the set of stepwise motions emanated from the point $\bar{x}$ under the strategy $\mathcal{A}$ applied in a discrete control scheme with the time partition Δ. Each of them obeys the differential equation

$$\dot{x}(t) = f\big(t, x(t), \mathcal{A}\big(x(t_i)\big), b(t)\big),\ t_i \le t < t_{i+1},\ i \in \mathbb{N}, \tag{6}$$

and the initial condition $x(0) = \bar{x}$ for some measurable second player's control realization $b(\cdot) : [0,\infty) \to B$. The elements of this set are absolutely continuous functions $x(\cdot) : [0,\infty) \to \mathbb{R}^n$. Define

$$\operatorname{diam}\Delta = \sup_{i\in\mathbb{N}}(t_{i+1} - t_i).$$

Take a sequence of points $\bar{x}_l \to x_0$ as $l \to \infty$, a sequence of partitions Δ_l such that $\operatorname{diam}\Delta_l \to 0$ as $l \to \infty$, and a sequence $x_l(\cdot) \in \mathbb{X}(x_{0,l}, \mathcal{A}, \Delta_l)$ of stepwise motions. If $\big\{x_l(\cdot)\big\}$ uniformly tends to some function $x_*(\cdot)$ as $l \to \infty$, then this limit function $x_*(\cdot)$

is called a *constructive motion* generated by the strategy $\mathcal{A}$ [12, p. 33], [13, p. 107] and is considered as a trajectory of the system. The set of all constructive motions emanated from the point x_0 under the first player's strategy $\mathcal{A}$ is denoted by $\mathbb{X}(x_0, \mathcal{A})$. The set $\mathbb{X}(x_0, \mathcal{A})$ is non-empty and sequentially compact in $C\big([0,\infty);\mathbb{R}^n\big)$, that is, from any sequence $x_k(\cdot) \in \mathbb{X}(x_0, \mathcal{A})$, $k \in \mathbb{N}$, one can select a converging subsequence $x_{k_l}(\cdot)$, $l \in \mathbb{N}$, which limit belongs to the set $\mathbb{X}(x_0, \mathcal{A})$. Here, the convergence is taken in the sense of compact-open topology (but not in the sense of ordinary metric of the space C, because the time interval is not bounded).

Similarly, we define the set $\mathbb{X}(\bar{x}, \mathcal{B}, \Delta)$ of stepwise motions of the system emanated from the point x_0 under the second player's strategy $\mathcal{B}$ in the discrete control scheme with the time partition Δ and the set $\mathbb{X}(x_0, \mathcal{B})$ of constructive motions generated by a feedback strategy $\mathcal{B} : \mathbb{R}^n \to B$ of the evader. For all strategies $\mathcal{B}$, the set $\mathbb{X}(x_0, \mathcal{B})$ is non-empty and sequentially compact in $C\big([0,\infty);\mathbb{R}^n\big)$ (in the compact-open topology).

The *guaranteed result $T_1^0(x_0)$ of the first player at the point x_0* is defined as follows:

$$\begin{aligned} T_1^\varepsilon(x_0,\mathcal{A}) &:= \sup\Big\{\tau_\varepsilon\big(x(\cdot)\big) : x(\cdot) \in \mathbb{X}(x_0,\mathcal{A})\Big\}, \\ T_1^\varepsilon(x_0) &:= \inf_{\mathcal{A}\in\mathbb{A}} T_1^\varepsilon(x_0,\mathcal{A}), \qquad T_1^0(x_0) := \lim_{\varepsilon\downarrow 0} T_1^\varepsilon(x_0). \end{aligned} \tag{7}$$

Here and below, $\mathbb{A}$ is the set of all feedback strategies of the first player.

In the same way, the *guaranteed result $T_2^0(x_0)$ of the second player at the point x_0* is defined as

$$\begin{aligned} T_2^\varepsilon(x_0,\mathcal{B}) &:= \inf\Big\{\tau_\varepsilon\big(x(\cdot)\big) : x(\cdot) \in \mathbb{X}(x_0,\mathcal{B})\Big\}, \\ T_2^\varepsilon(x_0) &:= \sup_{\mathcal{B}\in\mathbb{B}} T_2^\varepsilon(x_0,\mathcal{B}), \qquad T_2^0(x_0) := \lim_{\varepsilon\downarrow 0} T_2^\varepsilon(x_0). \end{aligned} \tag{8}$$

Here and below, $\mathbb{B}$ is the set of all feedback strategies of the second player.

The following inequalities hold for arbitrary discrete control schemes $(\mathcal{A}, \Delta^{(1)})$ and $(\mathcal{B}, \Delta^{(2)})$

$$T_1^\varepsilon(x_0,\mathcal{A},\Delta^{(1)}) \geq \tau_\varepsilon\big(x(\cdot)\big) \geq T_2^\varepsilon(x_0,\mathcal{B},\Delta^{(2)}), \tag{9}$$

for any

$$x(\cdot) \in \mathbb{X}(x_0,\mathcal{A},\Delta^{(1)}) \bigcap \mathbb{X}(x_0,\mathcal{B},\Delta^{(2)}).$$

Hence, $T_1^0(x_0) \geq T_2^0(x_0)$. Non-emptiness of intersection of the sets of motions is proved in [12,13,16]. Also, the results of books [12,13] imply the following.

Theorem 1 *For all initial positions $x_0 \in \mathbb{R}^n$ and for all sufficiently small $\varepsilon > 0$ in the time-optimal differential game* (1), (5) *with lifeline, there exists an instant $\omega^\varepsilon \in [0,\infty]$ such that $T_1^\varepsilon(x_0) = T_2^\varepsilon(x_0) = \omega^\varepsilon$.*

Using this result, we can prove the following.

Theorem 2 *In the time-optimal differential game* (1), (2) *with lifeline, the value function $T : \mathrm{cl}\mathcal{G} \to \mathbb{R} \cup \{+\infty\}$ exists that is $T(x_0) := T_1^0(x_0) = T_2^0(x_0)$ for all $x_0 \in \mathbb{R}^n$.*

Proof Let us consider the limits of the guaranteed results in ε:

$$\begin{aligned} T_1^\varepsilon(x_0) &= \inf_{\mathcal{A}\in\mathbb{A}} T_1^\varepsilon(x_0,\mathcal{A}), \qquad T_2^\varepsilon(x_0) = \sup_{\mathcal{B}\in\mathbb{B}} T_2^\varepsilon(x_0,\mathcal{B}) \\ T_1^0(x_0) &= \lim_{\varepsilon\to 0} T_1^\varepsilon(x_0), \qquad T_2^0(x_0) = \lim_{\varepsilon\to 0} T_2^\varepsilon(x_0). \end{aligned}$$

Let x_0 be such a point that for some $\varepsilon > 0$ and a strategy $\mathcal{A}$ of the first player, all trajectories from the set $\mathbb{X}(x_0, \mathcal{A})$ reach the set $\mathcal{T}_\varepsilon = \mathcal{T}$ remaining in the set cl $\mathcal{G}_\varepsilon$. Here and below, the symbol "cl" denotes the closure of a set. Then, for all $\varepsilon_1, \varepsilon_2$ such that $0 < \varepsilon_1 < \varepsilon_2 \leq \varepsilon$, we obtain $T_1^{\varepsilon_1}(x_0) = T_1^{\varepsilon_2}(x_0)$. If, however, the point x_0 is such that for all $\varepsilon > 0$ and strategy $\mathcal{A}$ of the first player, there is a trajectory in the set $\mathbb{X}(x_0, \mathcal{A})$, which either reaches the set $\mathcal{F}_\varepsilon$ before hitting the set $\mathcal{T}_\varepsilon$, or forever stays in the set $\mathcal{G}_\varepsilon$ while missing the set $\mathcal{T}_\varepsilon$, then we have that for all ε the result $T_1^\varepsilon(x_0) = +\infty$. Consequently, magnitudes of the functions from the family $\left\{T_1^\varepsilon(\cdot)\right\}_\varepsilon$ at any point x induce a numeric sequence, which finally stabilizes at some finite or infinite value when $\varepsilon \to 0$. Therefore, there is a pointwise limit $T_1^\varepsilon(\cdot) \to T_1^0(\cdot)$ as $\varepsilon \to 0$. Since for all ε, the functions $T_1^\varepsilon(\cdot)$ and $T_2^\varepsilon(\cdot)$ coincide, then also there exists a pointwise limit $T_2^0(\cdot)$ as $\varepsilon \to 0$ of functions $T_2^\varepsilon(\cdot)$, which coincides with $T_1^0(\cdot)$. This common limit is an ε-saddle point T of game (1), (2). The term *ε-saddle point* is thought of as each player can choose its strategy, which guarantees the result arbitrarily close to the value of T. By this means, the function $T : \mathbb{R}^n \to \mathbb{R}$ is the value function of game (1), (2). □

The unboundedness of the value function and cost functional can cause some uneasiness in numerical research of game (1), (2). For this reason, one often substitutes the unbounded cost functional with a bounded one by means of *Kruzhkov's transform*:

$$J\big(x(\cdot; x_0)\big) = \begin{cases} 1 - \exp\big(-\tau\big(x(\cdot; x_0)\big)\big), & \text{if } \tau < +\infty, \\ 1, & \text{otherwise.} \end{cases} \tag{10}$$

In such a case, the value function also becomes bounded and its magnitude belongs to the range from zero to unity.

4 Minimax Solution

4.1 Preliminary Definitions and Statements

This section uses the following definitions and statements from book [16] necessary hereafter.

Consider a boundary value problem for HJE:

$$H(x, Dv) - v = 0, \ \ x \in \Omega, \tag{11}$$

$$v(x) = \sigma(x), \ \ x \in \partial\Omega. \tag{12}$$

Here, Ω is a domain in the space $\mathbb{R}^n$; Dv denotes the gradient of the function v.

Definition 1 Let us consider variables

$$d^-v(x; g) = \liminf_{\varepsilon\to 0} \left\{\delta^{-1}\big(v(x+\delta g') - v(x)\big) : \delta \in (0, \varepsilon), \|g - g'\| \leq \varepsilon\right\},$$

$$d^+v(x; g) = \limsup_{\varepsilon\to 0} \left\{\delta^{-1}\big(v(x+\delta g') - v(x)\big) : \delta \in (0, \varepsilon), \|g - g'\| \leq \varepsilon\right\}.$$

These relations define the *lower and upper derivatives* of the function v at the point $x \in \Omega$ at the direction $g \in \mathbb{R}^n$.

Definition 2 Sets $D^-v(x)$ and $D^+v(x)$ defined by equalities

$$D^-v(x) = \left\{s \in \mathbb{R}^n : \langle s, g\rangle - d^-v(x; g) \leq 0, \ \forall\, g \in \mathbb{R}^n\right\},$$

$$D^+v(x) = \left\{s \in \mathbb{R}^n : \langle s, g\rangle - d^+v(x; g) \geq 0, \ \forall\, g \in \mathbb{R}^n\right\},$$

are called the *subdifferential* and *superdifferential* of the function v at the point $x \in \Omega$.

Definition 3 A *supersolution (subsolution) of equation* (11) is a lower semicontinuous (upper semicontinuous) function $v : \Omega \mapsto \mathbb{R}$ satisfying the condition $H(x, s) - v(x) \leq 0$ for all $x \in \Omega$ and $s \in D^{-}v(x)$ $(H(x, s) - v(x) \geq 0$ for all $x \in \Omega$ and $s \in D^{+}v(x))$.

Definition 4 A *supersolution of problem* (11), (12) is a lower semicontinuous function v : cl $\Omega \mapsto \mathbb{R}$ satisfying the following conditions:

(i) A restriction of the function v to the set Ω is a supersolution of the equation (11).
(ii) This function satisfies boundary condition (12) and is bounded in cl Ω.

Definition 5 A *subsolution of problem* (11), (12) is an upper semicontinuous function v : cl $\Omega \mapsto \mathbb{R}$ satisfying the following conditions:

(j) A restriction of the function v to the set Ω is a subsolution of the equation (11).
(jj) This function satisfies boundary condition (12) and is bounded in cl Ω.
(jjj) The function v is continuous at every point $x \in \partial\Omega$.

Definition 6 A *minimax solution of problem* (11), (12) is a function v : cl $\Omega \mapsto \mathbb{R}$ satisfying the relations

$$\lim_{k\to\infty} v^{(k)}(x) = v(x) = \lim_{k\to\infty} u_k(x), \qquad x \in \operatorname{cl}\Omega,$$

where $\{v^{(k)}\}_1^\infty$ $(\{u_k\}_1^\infty)$ is a sequence of supersolutions (subsolutions) of problem (11), (12).

We set forth the formulation of the theorem of minimax solution existence [16, § 18.6, p.224].

Theorem 3 *Assume that the following conditions hold:*

H.1 *The function $H(\cdot, 0)$ is bounded, that is, for all $x \in \Omega$ $\big|H(x, 0)\big| \leq c$.*
H.2 *The inequality $\big|H(x, p_1) - H(x, p_2)\big| \leq \rho(x)\|p_1 - p_2\|$ holds for all $x \in \Omega$ and p_1, $p_2 \in \mathbb{R}^n$; here, $\rho(x) = \big(1 + \|x\|\big)\mu$ and μ is some positive number.*
H.3 *For any bounded set $\Delta \subset \Omega$, there exists a constant $L(\Delta)$ such that the inequality $\big|H(x_1, p) - H(x_2, p)\big| \leq L(\Delta)\|x_1 - x_2\|(1 + \|p\|)$ holds for all $p \in \mathbb{R}^n$ and $x_1, x_2 \in \Delta$.*
H.4 *The boundary condition function $\sigma(\cdot)$ is continuous and bounded in $\partial\Omega$.*

And also let a subsolution of problem (11), (12) *exist in the sense of Definition* 5*. Then, there exists a unique minimax solution of problem* (11), (12)*. This minimax solution coincides with the minimal supersolution.*

It should be noted that when we deal with a semicontinuous subsolution of problem (11), (12), one can change Definitions 4 and 5 and formulate Theorem 4, where existence of a minimax *supersolution* is required.

Definition 7 A supersolution of problem (11), (12) is a lower semicontinuous function v : cl $\Omega \mapsto \mathbb{R}$ satisfying the following conditions:

(i) A restriction of the function v to the set Ω is a supersolution of Equation (11).
(ii) This function satisfies the boundary condition (12) and is bounded in cl Ω.
(iii) The function v is continuous at every point $x \in \partial\Omega$.

Definition 8 A subsolution of problem (11), (12) is an upper semicontinuous function v : cl $\Omega \mapsto \mathbb{R}$ satisfying the following conditions:

(j) A restriction of the function v to the set Ω is a subsolution of equation (11).

(jj) This function satisfies boundary condition (12) and is bounded in cl Ω.

Note that in comparison with Definitions 4 and 5, the continuity condition is moved from the definition of a subsolution to the definition of a supersolution.

Theorem 4 ([16], p. 255) *We assume that conditions H.1–H.3 hold, and a supersolution of problem* (11)*,* (12) *exists in the sense of Definition* 7*. Then, there exists a continuous minimax solution of problem* (11)*,* (12)*.*

4.2 Minimax Solution of HJE Corresponding to Time-Optimal Problem with Lifeline

Let us consider an HJE boundary value problem corresponding to time-optimal game (1), (10) with lifeline:

$$H\big(x, Du(x)\big) - u(x) = 0, \ \ x \in \mathcal{G}, \tag{13}$$

$$u(x) = 0 \text{ if } x \in \partial\mathcal{T}, \quad u(x) = 1 \text{ if } x \in \partial\mathcal{F}, \tag{14}$$

where $H(x, s) = \mathcal{H}(x, s) + 1$; the function $\mathcal{H}(x, s)$ is the Hamiltonian defined by (4).

The aim of this section is to show that this problem has a minimax solution.

Proposition 1 *Let Conditions C.1–C.3, and C.5 hold. Then, there exists a function $\underline{u}(x)$, which is a subsolution of problem* (13)*,* (14) *in the sense of Definition* 5*.*

Proof We choose and fix a number $\varepsilon > 0$ and extend the set $\mathcal{F}$ by a ball of radius ε: $\mathcal{F}_\varepsilon = \mathcal{F} + B_\varepsilon$, $\mathcal{G}_\varepsilon = \mathbb{R}^n \setminus (\mathcal{F}_\varepsilon \cup \mathcal{T})$ (see Fig. 2). Again, we suppose that the number ε is sufficiently small so that the set $\mathcal{G}_\varepsilon$ is non-empty and the boundary $\partial\mathcal{F}_\varepsilon$ is smooth.

We construct the function $\underline{u} : \mathbb{R}^n \to \mathbb{R}$ in the following manner:

$$\underline{u}(x) := \begin{cases} 0, & x \in \mathcal{G}_\varepsilon \cup \partial\mathcal{T}, \\ 1 - \frac{\operatorname{dist}(x,\mathcal{F})}{\varepsilon}, & x \in \mathcal{F}_\varepsilon \setminus \operatorname{int}\mathcal{F}, \end{cases} \tag{15}$$

where int $\mathcal{F}$ is the interior of the set $\mathcal{F}$.

The constructed function is continuous, and, therefore, it is upper semi-continuous; hence, conditions (jj) and (jjj) from Definition 5 of the subsolution of the boundary value problem hold. Moreover, because the boundary of the set $\mathcal{F}$ is continuously differentiable, one can choose a number ε so small that the function $\underline{u}$ is continuously differentiable at the set

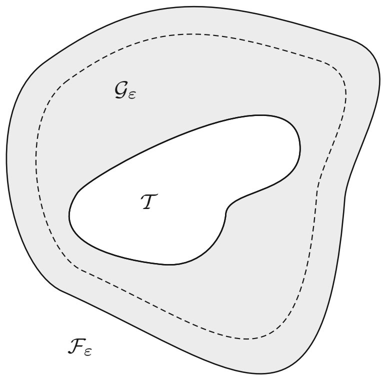

Fig. 2 Sets $\mathcal{T}$, $\mathcal{F}_\varepsilon$ and $\mathcal{G}_\varepsilon$. The outer solid line is $\partial\mathcal{F}$, and the dashed line is $\partial\mathcal{F}_\varepsilon$

$\mathcal{F}_\varepsilon \setminus \operatorname{int} \mathcal{F}$ (also at the boundary of this set). We assume that the number ε satisfies this condition.

Thus, one needs to prove that a restriction of the function $\underline{u}$ to the set $\mathcal{G}$ is a subsolution of problem (13). For this purpose, we are going to use Definition 3:

$$H(x, s) - \underline{u}(x) \geq 0, \qquad \forall s \in D^+\underline{u}(x).$$

Let us consider three cases:

(i) Let $x \in \mathcal{G}_\varepsilon$. Then $\underline{u}(x) = 0$, $D^+\underline{u}(x) = \{\mathbf{0}\}$.

$$H(x, s) - \underline{u}(x) = \min_{a\in A} \max_{b\in B} \big\langle \mathbf{0}, f(x, a, b)\big\rangle + 1 - 0 = 1 \geq 0.$$

Here and below, $\mathbf{0}$ is the origin of the corresponding space.

(ii) Let $x \in \operatorname{int} \mathcal{F}_\varepsilon$. We assume that the point $x_0 \in \partial\mathcal{F}$ is such that the shortcut from the point x to the set $\mathcal{F}$ is $\operatorname{dist}(x, \mathcal{F}) := \|x - x_0\|$.

Then, by reason of continuous differentiability of the boundary of the set $\mathcal{F}$ and of the function $\underline{u}$ at the set $\operatorname{int} \mathcal{F}_\varepsilon \setminus \mathcal{F}$, we get

$$x - x_0 = n_{\mathcal{F}}(x_0)\|x - x_0\| = n_{\mathcal{F}}(x_0) \operatorname{dist}(x, \mathcal{F}).$$

In this point, the function $\underline{u}(x) = 1 - \varepsilon^{-1} \operatorname{dist}(x, \mathcal{F})$ and

$$D^+\underline{u}(x) = \big\{\nabla\underline{u}(x)\big\}, \quad \nabla\underline{u}(x) = -\big(\varepsilon \operatorname{dist}(x, \mathcal{F})\big)^{-1} \cdot (x - x_0) = -\varepsilon^{-1} n_{\mathcal{F}}(x_0).$$

Now we estimate the expression $H(x, s) - \underline{u}(x)$:

$$H(x, s) - \underline{u}(x) = \min_{a\in A} \max_{b\in B} \big\langle \nabla\underline{u}(x), f(x, a, b)\big\rangle + 1 - 1 + \frac{\operatorname{dist}(x, \mathcal{F})}{\varepsilon}$$

(add and subtract $f(x_0, a, b)$)

$$\begin{aligned}
&= \frac{1}{\varepsilon} \min_{a\in A} \max_{b\in B} \big\langle - n_{\mathcal{F}}(x_0), f(x, a, b) + f(x_0, a, b) - f(x_0, a, b)\big\rangle + \frac{\operatorname{dist}(x, \mathcal{F})}{\varepsilon} \\
&= \frac{1}{\varepsilon} \min_{a\in A} \max_{b\in B} \Big\{\big\langle - n_{\mathcal{F}}(x_0), f(x_0, a, b)\big\rangle - \big\langle n_{\mathcal{F}}(x_0), f(x, a, b) - f(x_0, a, b)\big\rangle\Big\} \\
&\quad + \frac{\operatorname{dist}(x, \mathcal{F})}{\varepsilon} \\
&\geq \frac{1}{\varepsilon} \min_{a\in A} \max_{b\in B} \Big\{\big\langle - n_{\mathcal{F}}(x_0), f(x_0, a, b)\big\rangle - \|n_{\mathcal{F}}(x_0)\| \cdot \|f(x, a, b) - f(x_0, a, b)\|\Big\} \\
&\quad + \frac{\operatorname{dist}(x, \mathcal{F})}{\varepsilon} \\
&\geq \frac{1}{\varepsilon} \min_{a\in A} \max_{b\in B} \big\langle - n_{\mathcal{F}}(x_0), f(x_0, a, b)\big\rangle - \frac{\lambda\varepsilon}{\varepsilon} + \frac{\operatorname{dist}(x, \mathcal{F})}{\varepsilon} \\
&\geq \frac{1}{\varepsilon} \min_{a\in A} \max_{b\in B} \big\langle - n_{\mathcal{F}}(x_0), f(x_0, a, b)\big\rangle - \lambda + \frac{\operatorname{dist}(x, \mathcal{F})}{\varepsilon} \\
&= \frac{1}{\varepsilon}\Big(\min_{a\in A} \max_{b\in B} \big\langle - n_{\mathcal{F}}(x_0), f(x_0, a, b)\big\rangle + \operatorname{dist}(x, \mathcal{F})\Big) - \lambda.
\end{aligned}$$

The quantity λ is the Lipschitz constant of the function f w.r.t. the argument x from Condition C.1 [from relation (3)].

Due to Condition C.5, continuity of the function f, continuity of the vector $n_{\mathcal{F}}(x)$ along the boundary of the set $\mathcal{F}$ (because of the continuous differentiability and compactness of the boundary of the set $\mathcal{F}$), and continuity of the scalar product on its arguments, one can state that there exists such a number $\eta > 0$ that for any point $x_0 \in \partial\mathcal{F}$ the inequality holds:

$$\min_{a\in A}\max_{b\in B}\big\langle -n_{\mathcal{F}}(x_0),\, f(x_0,a,b)\big\rangle \geq \eta > 0.$$

Then, on the basis of the number η, one can choose such a number ε that

$$\frac{1}{\varepsilon}\Big(\min_{a\in A}\max_{b\in B}\big\langle -n_{\mathcal{F}}(x_0),\, f(x_0,a,b)\big\rangle + \operatorname{dist}(x,\mathcal{F})\Big) - \lambda \geq 0,$$

and fulfillment of this inequality implies fulfillment of the desired one.

(iii) Let $x \in \partial\mathcal{F}_\varepsilon$. One has $\underline{u}(x) = 0$. The point x belongs to the smooth line $\partial\mathcal{F}_\varepsilon$, where two smooth branches of the function $\underline{u}(\cdot)$ join. Therefore, there are two partial limits of $\nabla\underline{u}(y)$ as $y \to x$: the zero vector (if $y \in \mathcal{G}_\varepsilon$) and some vector $\kappa(x)$ (if $y \in \mathcal{F}_\varepsilon$). Because the function $\underline{u}(\cdot)$ has two smooth branches at the point x, one has that $D^+\underline{u}(x)$ is the convex hull of the set of these two partial limits of $\nabla\underline{u}(\cdot)$ at the point x: $D^+\underline{u}(x) = \operatorname{co}\big\{\kappa(x), \mathbf{0}\big\} = \big\{\gamma\kappa(x) : \gamma \in [0,1]\big\}$. The sign "co" denotes the convex hull of the set argument.

Let us make a similar estimate:

$$\begin{aligned} H(x,s) - \underline{u}(x) &= \min_{a\in A}\max_{b\in B}\big\langle \gamma\kappa(x),\, f(x,a,b)\big\rangle + 1 - 0 \\ &= \frac{1}{\varepsilon}\gamma\min_{a\in A}\max_{b\in B}\big\langle -n_{\mathcal{F}}(x_0),\, f(x,a,b)\big\rangle + 1. \end{aligned}$$

Similarly to the previous case, we obtain the following inequality by means of the same Condition C.5:

$$H(x,s) - \underline{u}(x) \geq \gamma\left(\frac{1}{\varepsilon}\min_{a\in A}\max_{b\in B}\big\langle -n_{\mathcal{F}}(x_0),\, f(x_0,a,b)\big\rangle - \lambda\right) + 1,$$

the right-hand side of which can be made greater than zero for all $\gamma \in [0,1]$ by choosing the number ε.

Thus, the inequality from Definition 3 is proved for all points from the set $\mathcal{G}$; therefore, the function $\underline{u}$ is a subsolution of equation (13). And since the function $\underline{u}$ satisfies the boundary condition and is continuous at $\partial\mathcal{G}$, it is a subsolution of problem (13), (14) in the sense of Definition 5. □

Thus, under Condition C.5, problem (13), (14) has a subsolution in the sense of Definition 5, and the premises of Theorem 3 hold. Consequently, under this assumption, problem (13), (14) has a minimax solution.

Now we are going to check that the premises of Theorem 4 hold, where the existence of a supersolution of problem (13), (14) in the sense of Definition 7 is required (that is, with the additional assumption on the continuity of the supersolution on the boundary of the set $\partial\mathcal{G}$).

It is worthwhile to say that the existence of the minimax solution already does not require an accomplishment of Theorem 4. Nevertheless, subsequently, the existence of a continuous supersolution will be necessary; let us prove it.

Proposition 2 *Let Conditions C.1–C.3, and C.4 hold. Then, there exists a function* $\bar{u}(x)$*, which is a supersolution of problem* (13)*,* (14) *in the sense of Definition* 7.

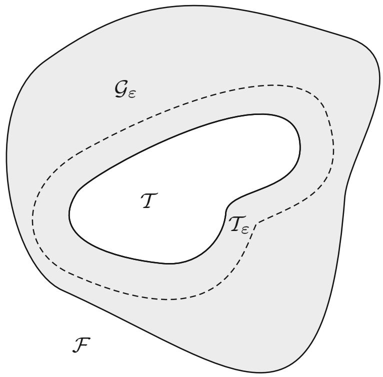

Fig. 3 Sets $\mathcal{T}_\varepsilon$, $\mathcal{G}_\varepsilon$ and $\mathcal{F}$. The inner solid line is $\partial\mathcal{T}$, and the dashed line is $\partial\mathcal{T}_\varepsilon$

Proof In the similar way as while constructing the subsolution, we extend the set $\mathcal{T}$ by a ball with some small radius $\varepsilon > 0$ to get the sets $\mathcal{T}_\varepsilon := \mathcal{T} + B_\varepsilon$, $\mathcal{G}_\varepsilon = \mathbb{R}^n \setminus (\mathcal{F} \cup \mathcal{T}_\varepsilon)$ (see Fig. 3).

We define a function $\bar{u} : \operatorname{cl}\mathcal{G} \mapsto \mathbb{R}$ in such a way:

$$\bar{u}(x) := \begin{cases} 1, & x \in \mathcal{G}_\varepsilon \cup \partial\mathcal{F}, \\ \frac{\operatorname{dist}(x,\mathcal{T})}{\varepsilon}, & x \in \mathcal{T}_\varepsilon \setminus \operatorname{int}\mathcal{T}. \end{cases}$$

The function $\bar{u}$ is bounded, continuous on the boundary of the set $\mathcal{G}$, and it satisfies the boundary condition. In virtue of the continuous differentiability of the boundary of the set $\mathcal{G}$, one can choose sufficiently small ε so that the function $\bar{u}$ is continuously differentiable at the set $\mathcal{T}_\varepsilon \setminus \operatorname{int}\mathcal{T}$ and the boundary $\partial\mathcal{T}_\varepsilon$ is smooth. We suppose that such an ε is chosen.

Let us check that $\bar{u}$ is a supersolution of problem (13). By Definition 3, the function $\bar{u}(x)$ is a supersolution if the following inequality holds:

$$H(x, s) - \bar{u}(x) \le 0, \quad \forall\, x \in \mathcal{G}, \quad \forall\, s \in D^- u(x).$$

We examine three cases:

(i) Let $x \in \mathcal{G}_\varepsilon$. Then $\bar{u}(x) = 1$, $D^-\bar{u}(x) = \{\mathbf{0}\}$.

$$H(x, s) - \bar{u}(x) = \min_{a\in A} \max_{b\in B} \big\langle \mathbf{0},\, f(x, a, b)\big\rangle + 1 - 1 = 0 \le 0.$$

(ii) Let $x \in \operatorname{int}\mathcal{T}_\varepsilon$. Then, $\bar{u}(x) = \varepsilon^{-1}\operatorname{dist}(x, \mathcal{T})$ at the given point; let shortcut from a point x to the set $\mathcal{T}$ is attained at a point $x_0 \in \partial\mathcal{T}$, that is, $\operatorname{dist}(x, \mathcal{T}) := \|x - x_0\|$.

$$D^-\bar{u}(x) = \big\{\nabla\bar{u}(x)\big\}, \quad \nabla\bar{u}(x) = \big(\varepsilon\operatorname{dist}(x, \mathcal{T})\big)^{-1} \cdot (x - x_0) = \varepsilon^{-1} n_{\mathcal{T}}(x_0).$$

Let us estimate the difference $H(x, s) - \bar{u}(x)$:

$$H(x, s) - \bar{u}(x) = \min_{a\in A} \max_{b\in B} \big\langle \nabla\bar{u}(x),\, f(x, a, b)\big\rangle + 1 - \frac{\operatorname{dist}(x, \mathcal{T})}{\varepsilon}$$

(add and subtract $f(x_0, a, b)$)

$$
\begin{aligned}
&= \frac{1}{\varepsilon} \min_{a\in A} \max_{b\in B} \big\langle n_{\mathcal{T}}(x_0),\, f(x,a,b) + f(x_0,a,b) - f(x_0,a,b)\big\rangle + 1 - \frac{\operatorname{dist}(x,\mathcal{T})}{\varepsilon} \\
&\le \frac{1}{\varepsilon} \min_{a\in A} \Big\{ \max_{b\in B} \big\langle n_{\mathcal{T}}(x_0),\, f(x_0,a,b)\big\rangle + \max_{b\in B} \big\langle n_{\mathcal{T}}(x_0),\, f(x,a,b) - f(x_0,a,b)\big\rangle \Big\} \\
&\quad + 1 - \frac{\operatorname{dist}(x,\mathcal{T})}{\varepsilon} \\
&\le \frac{1}{\varepsilon} \min_{a\in A} \Big\{ \max_{b\in B} \big\langle n_{\mathcal{T}}(x_0),\, f(x_0,a,b)\big\rangle + \max_{b\in B} \|n_{\mathcal{T}}(x_0)\| \cdot \|f(x,a,b) - f(x_0,a,b)\| \Big\} \\
&\quad + 1 - \frac{\operatorname{dist}(x,\mathcal{T})}{\varepsilon} \\
&= \frac{1}{\varepsilon} \min_{a\in A} \max_{b\in B} \big\langle n_{\mathcal{T}}(x_0),\, f(x_0,a,b)\big\rangle + \frac{1}{\varepsilon}\|n_{\mathcal{T}}(x_0)\| \cdot \lambda \|x - x_0\| + 1 - \frac{\operatorname{dist}(x,\mathcal{T})}{\varepsilon} \\
&\le \frac{1}{\varepsilon} \min_{a\in A} \max_{b\in B} \big\langle n_{\mathcal{T}}(x_0),\, f(x_0,a,b)\big\rangle + \frac{\lambda\varepsilon}{\varepsilon} + 1 - \frac{\operatorname{dist}(x,\mathcal{T})}{\varepsilon} \\
&= \frac{1}{\varepsilon} \Big(\min_{a\in A} \max_{b\in B} \big\langle n_{\mathcal{T}}(x_0),\, f(x_0,a,b)\big\rangle - \operatorname{dist}(x,\mathcal{T}) \Big) + \lambda + 1.
\end{aligned}
$$

Due to Condition C.4, continuity of the function f, continuity of the vector $n_{\mathcal{T}}(x)$ along the boundary of the set $\mathcal{T}$ (by the reason of continuous differentiability of the boundary of the set $\mathcal{T}$), and continuity of the scalar product on all arguments, one can choose such a number $\eta < 0$ that for every point $x_0 \in \partial\mathcal{T}$ the following inequality is true:

$$\min_{a\in A} \max_{b\in B} \big\langle n_{\mathcal{T}}(x_0),\, f(x_0,a,b)\big\rangle \le \eta < 0.$$

Then, on the basis of the number η, one can take such a number ε that

$$\frac{1}{\varepsilon} \Big(\min_{a\in A} \max_{b\in B} \big\langle n_{\mathcal{T}}(x_0),\, f(x_0,a,b)\big\rangle - \operatorname{dist}(x,\mathcal{T}) \Big) + \lambda + 1 \le 0,$$

and fulfillment of this inequality implies fulfillment of the desired one.

(iii) Let $x \in \partial\mathcal{T}_\varepsilon$. One has $\bar{u}(x) = 1$. The point x belongs to the smooth line $\partial\mathcal{T}_\varepsilon$, where two smooth branches of the function $\bar{u}(\cdot)$ join. Therefore, there are two partial limits of $\nabla\bar{u}(y)$ as $y \to x$: the zero vector (if $y \in \mathcal{G}_\varepsilon$) and some vector $\kappa(x)$ (if $y \in \mathcal{T}_\varepsilon$). Because the function $\bar{u}(\cdot)$ has two smooth branches at the point x, one has that $D^-\bar{u}(x)$ is the convex hull of the set of these two partial limits of $\nabla\bar{u}(\cdot)$ at the point x: $D^-\bar{u}(x) = \operatorname{co}\big\{\kappa(x), \mathbf{0}\big\} = \big\{\gamma\kappa(x) : \gamma \in [0,1]\big\}$. Then,

$$
\begin{aligned}
H(x,s) - u(x) &= \min_{a\in A} \max_{b\in B} \big\langle \gamma\kappa(x),\, f(x,a,b)\big\rangle + 1 - 1 \\
&= \frac{1}{\varepsilon}\gamma \min_{a\in A} \max_{b\in B} \big\langle n_{\mathcal{T}}(x_0),\, f(x,a,b)\big\rangle.
\end{aligned}
$$

In the same way as in the previous case under Condition C.4 of the admissibility $\partial\mathcal{T}$, we reach the relation

$$H(x,s) - u(x) \le \gamma \Big(\frac{1}{\varepsilon} \min_{a\in A} \max_{b\in B} \big\langle n_{\mathcal{T}}(x_0),\, f(x_0,a,b)\big\rangle + \lambda \Big),$$

the right-hand side of which can be made less than zero for all $\gamma \in [0,1]$ with the help of choosing an appropriate number ε.

Thus, the inequality from Definition 3 is proved for all points from the set $\mathcal{G}$; therefore, the function $\bar{u}$ is a supersolution of equation (13). And since the function $\bar{u}$ satisfies the boundary condition and is continuous at $\partial\mathcal{G}$, it is a supersolution of problem (13), (14) in the sense of Definition 4. □

Therefore, if $\partial\mathcal{T}$ is an admissible zone (Condition C.4 holds), then problem (13), (14) has a supersolution in the sense of Definition 7 and Theorem 4 holds. That is, under this assumption, problem (13), (14) has a minimax solution.

As a result, it can be said that problem (13), (14) has a minimax solution if at least one of Conditions C.4 and C.5 holds. Note that the accomplishment of Condition C.5 results in the continuity of the value function nearby $\partial\mathcal{F}$ and Condition C.4 provides continuity of the value function near $\partial\mathcal{T}$. If both conditions hold (as it will be required below in Sects. 5.1 and 5.2), then the value function is continuous on the set $\mathcal{G}$. Of special interest is a proof of existence of a minimax solution of problem (13), (14) under some weaker assumptions.

5 Coincidence of Value Function and Minimax Solution

In book [16], the coincidence of the value function of a classic time-optimal problem and the minimax solution of a boundary value problem of the corresponding HJE is proved in the following way. At first, one proves that for any constant greater than the magnitude of the minimax solution (recomputed by the inverse Kruzhkov's transform), the guaranteed result of the first player can be done less than this constant by choice of the strategy $\mathcal{A}$ of the first player and the time partition Δ of the discrete control scheme; it is Theorem 19.6 in Subsection 19.6 of book [16]. Further, it is proved that for any constant less than the magnitude of the minimax solution (recomputed by the inverse Kruzhkov's transform), the guaranteed result of the second player can be done greater than this constant by choice of the strategy $\mathcal{B}$ of the second player and time partition Δ of the discrete control scheme; it is Theorem 19.8 in Subsection 19.8 of book [16]. Finally, it is concluded that the guaranteed results converge to the magnitude of the minimax solution. This together with their convergence to the value function gives the coincidence.

Theorems 19.6 and 19.8 are not absolutely symmetric. Whereas Theorem 19.6 uses the notion of a minimax solution, a subsolution is used in Theorem 19.8. This dissymmetry is because of different aims of the players. So, in the classic situation, whereas the aim of the first player is to lead the trajectory of the system to the target set, the aim of the second player is to avoid an ε-neighborhood of this set.

Now, let us formulate and prove analogous theorems for the time-optimal problem with lifeline. It should be noted that in Theorem 6 a supersolution is used instead of minimax solution, because the first player tries not only to lead the system to the target $\mathcal{T}$, but also to avoid an ε-neighborhood of the set $\mathcal{F}$.

Firstly, we prove a theorem on the estimate of guaranteed result of the second player for a time-optimal problem with lifeline.

5.1 Estimate of Guaranteed Result of the Second Player

Let $u : \operatorname{cl}\mathcal{G} \mapsto [0, 1]$ be a minimax solution (in sense of Definition 6) of problem (13), (14), and $u_\natural$ be a subsolution (in sense of Definition 5) of this problem. We introduce the following transform of the function $u_\natural$:

$$u_\natural^\alpha(x) := \max_{y\in \mathrm{cl}\,\mathcal{G}} \{u_\natural(y) - w_\alpha(x, y)\},$$

where

$$w_\alpha(x, y) := \frac{(\alpha^{2/\nu} + \|x-y\|^2)^\nu}{\alpha},$$
$$\nu := (2+2\lambda)^{-1}, \qquad 0 < \alpha < \min\{1/3, [\lambda(1+\lambda)]^{-1}\}, \tag{16}$$

and λ is the coefficient from Lipschitz condition (3).

We choose

$$y_\alpha(x) \in \mathrm{Arg}\max_{y\in \mathrm{cl}\,\mathcal{G}} [u_\natural(y) - w_\alpha(x, y)].$$

We define the feedback strategy $\mathcal{B}_\alpha : \mathrm{cl}\,\mathcal{G} \mapsto B$ of the second player by means of the relation

$$\mathcal{B}_\alpha(x) = b_0(x, s_\alpha(x)), \tag{17}$$

where b_0 is an extremal pre-strategy defined by the equality

$$b_0(x, s) \in \mathrm{Arg}\max_{b\in B} \Big[\min_{a\in A} \langle s, f(x, a, b)\rangle\Big],$$

and the vector $s_\alpha(x)$ is given by the formula

$$s_\alpha(x) := -(D_x w_\alpha)(x, y_\alpha(x)) = (D_y w_\alpha)(x, y_\alpha(x)).$$

Theorem 5 *Let Conditions C.1–C.3 and C.5 hold; this guarantees existence and uniqueness of the minimax solution $u(\cdot) : \mathrm{cl}\,\mathcal{G} \mapsto [0, 1]$ of problem (13), (14). Let x_0 be a point in the set $\mathcal{G}$ such that $u(x_0) < 1$, and let θ be a positive number satisfying the inequality*

$$\theta < -\ln(1 - u(x_0)).$$

Then, there exists a subsolution $u_\natural$ of this problem satisfying the inequality

$$\theta < \omega_\natural := -\ln(1 - u_\natural(x_0)), \tag{18}$$

and such numbers $\varepsilon > 0$, $\alpha > 0$ that the estimate

$$T_2^\varepsilon(x_0, \mathcal{B}_\alpha) \geq \theta \tag{19}$$

holds. Here, $\mathcal{B}_\alpha$ is a feedback strategy of the second player defined by relation (17) and the variable $T_2^\varepsilon(x_0, \mathcal{B}_\alpha)$ is introduced by equality (8).

Proof By Proposition 1, we pick out a subsolution $u_\natural$ continuous on the boundary of $\mathcal{G}$ in such a way. The minimax solution exists because of the satisfaction of Condition C.5, which results in the existence of the sequence of subsolutions, which are continuous on the boundary of the set $\mathcal{G}$ and converge pointwisely to this minimax solution. Among the elements of the sequence, one can choose a subsolution so that inequality (18) holds.

We choose some numbers $\varepsilon > 0$ and $\alpha \in (0, \varepsilon/3]$. Also, we choose a sequence of points $\bar{x}_n \to x_0$ and a sequence of partitions $\{\Delta_n\}$ of the positive time semiaxis such that $\delta_n = \mathrm{diam}\,\Delta_n \to 0$. By the sequence of partitions $\{\Delta_n\}$, we construct a sequence of sets of constructive motions $\mathbb{X}(\bar{x}_n, \mathcal{B}_\alpha, \Delta_n) =: \mathbb{X}_n^\alpha$.

In a time-optimal problem with lifeline, three cases for a motion are possible: The trajectory reaches the set $\mathcal{T}$ not hitting the set $\mathcal{F}$, the trajectory stays in the set $\mathcal{G}$ forever, and the trajectory reaches the set $\mathcal{F}$ not touching the set $\mathcal{T}$. Thus, the proof is breaking apart into the following cases:

(I) There exist $\bar{\varepsilon} > 0$, $\bar{\alpha} \in (0, \bar{\varepsilon}/3]$ such that for all sequences $\{x_n(\cdot) \in \mathbb{X}_n^{\alpha}\}$, we have that for all $t \le \theta$ the inequality $\operatorname{dist}\big(x_n(t), \mathcal{T}\big) \ge 3\bar{\alpha}$ holds. Herewith, two cases are possible:

(I.1) For every n, there exists a trajectory $x_n(\cdot) \in \mathbb{X}_n^{\alpha}$, for which there exists an instant $t_n \in \Delta_n$ such that $t_n < \theta$ and $\operatorname{dist}(x_n(t_n), \mathcal{F}) \le 3\bar{\alpha}$.

(I.2) Alternatively, for all n and for every trajectory $x_n(\cdot) \in \mathbb{X}_n^{\alpha}$, the opposite inequality $\operatorname{dist}(x_n(t), \mathcal{F}) > 3\bar{\alpha}$ holds for all $t \in \Delta_n$ satisfying the restriction $t < \theta$.

(II) Conversely, let for all $\varepsilon > 0$ and $\alpha \in (0, \varepsilon/3]$ there exist a sequence $\{x_n(\cdot) \in \mathbb{X}_n^{\alpha}\}$ and a collection of instants $\bar{t}_n, \bar{t}_n < \theta$, which satisfy the inequality $\operatorname{dist}\big(x_n(\bar{t}_n), \mathcal{T}\big) < 3\alpha$.

In case (I), estimate (19) is obvious. Indeed, in situation (I.1), there exists a strategy of the second player, which leads every constructive motion of the system to the set $\mathcal{F}$ up to the instant θ, so this strategy leads every constructive motion from the set $\mathbb{X}(x_0, \mathcal{B}_{\bar{\alpha}})$ to the set $\mathcal{F}$ up to the instant θ. Thus, the guaranteed result of the second player is equal to $+\infty$, which is greater than θ. In situation (I.2), there exists a second player's strategy, which repels any stepwise motion of the system from some neighborhood of the terminal set up to the instant θ, so the second player can also keep all constructive motions away from the terminal set until the same instant, that is, $\forall\, n \in \mathbb{N}\;\; \tau_{\bar{\varepsilon}}\big(x_n(\cdot)\big) \ge \theta$ and so, $T_2^{\bar{\varepsilon}}(x_0, \mathcal{B}_{\bar{\alpha}}) \ge \theta$.

Now, let us show that case (II) gives a contradiction.

We take an arbitrary partition $\Delta = \{t_i\}$ of the time axis. We can prove a relation on change of the function $u_{\natural}^{\alpha}$ along the motion from the set $\mathbb{X}_n^{\alpha}$ analogous to that from the proof of Theorem 19.6 [16, § 19.6, pp. 248–250]:

$$u_{\natural}^{\alpha}\big(x(\tau)\big) \ge 1 - \big[1 - u_{\natural}^{\alpha}(x_{t_i})\big]e^{\tau - t_i} - (\tau - t_i)e^{\tau - t_i}h(\delta), \tag{20}$$

where $\tau \in [t_i, t_{i+1}] \cap [0, \theta_0]$. Here, δ is the diameter of the partition Δ, and $h(\delta) \to 0$ as $\delta \to 0$. The instant $\theta_0 = t_{k+1}$ is such that $t_k,\, t_{k+1} < \theta$, $\operatorname{dist}\big(x(t_k), \mathcal{T}\big) > 3\alpha$, and $\operatorname{dist}\big(x(t_{k+1}), \mathcal{T}\big) \le 3\alpha$. Summing (20) over all segments $[t_i, t_{i+1}]$, we get

$$\begin{aligned} u_{\natural}^{\alpha}\big(x(\theta_0)\big) &\ge 1 - \big[1 - u_{\natural}^{\alpha}(x_0)\big]e^{\theta_0} - \theta_0 e^{\theta_0}h(\delta) \\ &\ge 1 - e^{\theta_0 - \omega_{\natural}} - e^{\theta_0}\alpha - \theta_0 e^{\theta_0}h(\delta). \end{aligned} \tag{21}$$

We take arbitrary sequences $\{\varepsilon_n\} \to 0$ and $\big\{\alpha_n : \alpha_n \in (0, \varepsilon_n/3]\big\}$. By them, we construct a functional sequence $\{x_n(\cdot) : x_n(\cdot) \in \mathbb{X}_n^{\alpha_n}\}$ and a sequence of instants $\big\{\bar{t}_n \in \Delta_n : \operatorname{dist}\big(x(\bar{t}_n), \mathcal{T}\big) < 3\alpha_n\big\}$. On the basis of the numeric sequences $\{\alpha_n\}$ and $\{\bar{t}_n\}$ and the functional sequences $\big\{u_{\natural}^{\alpha_n}(\cdot)\big\}$ and $\big\{x_n(\cdot)\big\}$, we get the following numeric sequence:

$$\Big\{u_{\natural}^{\alpha_n}\big(x_n(\bar{t}_n)\big)\Big\}, \text{ where } u_{\natural}^{\alpha_n}(x) := \max_{y \in \operatorname{cl}\mathcal{G}}\big\{u_{\natural}(y) - w_{\alpha_n}(x, y)\big\}. \tag{22}$$

In the same manner as in the reasoning in Subsection 19.5 of book [16, § 19.5, p. 246], one can show that the maximum in (22) is achieved in such a point $y_{\alpha_n}(x)$ that $\big\|y_{\alpha_n}(x) - x\big\| \le 2\alpha_n$. Further, the parentheses of y_{α_n} with x inside will be omitted. It is obvious that

$$u_{\natural}^{\alpha_n}(x) \ge u_{\natural}(x) - w_{\alpha_n}(x, x) \ge -\alpha_n.$$

Then, $w_{\alpha_n}(x, y_{\alpha_n}) = u_{\natural}(y_{\alpha_n}) - u_{\natural}^{\alpha_n}(x) \le 1 + \alpha_n$. Hence,

$$\begin{aligned} &\Big[\alpha_n^{2/\nu} + \big\|x - y_{\alpha_n}(x)\big\|^2\Big]^{\nu} \le (1 + \alpha_n)\alpha_n, \\ &\big\|x - y_{\alpha_n}(x)\big\|^2 \le \big[(1 + \alpha_n)\alpha_n\big]^{1/\nu} - \alpha_n^{2/\nu} \le \big[(1 + \alpha_n)\alpha_n\big]^{1/\nu} \le 2\alpha_n^2. \end{aligned}$$

Therefore, $y_{\alpha_n}(x) \in O_{2\alpha_n}(x)$. Here and below, the symbol $O_\rho(x)$ stands for an open ball with the center at the point x and radius ρ. From this, it follows that

$$u_\natural^{\alpha_n}(x) := \max_{y \in O_{2\alpha_n}(x)} \{u_\natural(y) - w_{\alpha_n}(x, y)\}.$$

By the choice of the instants $\bar{t}_n$, they satisfy inequality (21) with substitution of θ_0 with $\bar{t}_n$:

$$\begin{aligned} u_\natural^{\alpha_n}(x_n(\bar{t}_n)) &\geq 1 - [1 - u_\natural^{\alpha_n}(x_0)]e^{\bar{t}_n} - \bar{t}_n e^{\bar{t}_n} h(\delta_n) \\ &\geq 1 - e^{\bar{t}_n - \omega_\natural} - \alpha_n e^{\bar{t}_n} - \bar{t}_n e^{\bar{t}_n} h(\delta_n). \end{aligned} \tag{23}$$

The sequence $\{\bar{t}_n\}$ is bounded, and consequently, one can select a convergent subsequence $\{\bar{t}_{n_k}\} \to \bar{t}$. In order not to overcharge the proof with double indexing, we assume that the sequence $\{\bar{t}_n\}$ itself converges. Since the inequality $\bar{t}_n < \theta$ holds for all n, then $\bar{t} \leq \theta$.

Now, let us consider the sequence $\{x_n(\cdot)\}$ in the interval $[0, \bar{t}]$. Generally speaking, a function $x_n(\cdot)$ can be defined in a smaller interval (if $\bar{t}_n < \bar{t}$), but it can be expanded continuously in a "good" manner to the entire interval $[0, \bar{t}]$, for example, $x_n(t) = x_n(\bar{t}_n)$, $t \in (\bar{t}_n, \bar{t}\,]$. The elements of the sequence are solutions of the differential inclusion, so $\{x_n(\cdot)\}$ is a uniformly bounded and equicontinuous sequence in the interval $[0, \bar{t}\,]$ (after a "good" expansion). Consequently, one can apply the Arzela–Ascoli theorem and get a uniformly convergent subsequence; let the very sequence $\{x_n(\cdot)\}$ be convergent. We denote its limit by $\bar{x}(\cdot)$.

By reason of the uniform convergence of the $x_n(\cdot) \rightrightarrows \bar{x}(\cdot)$, we obtain that the limit $\bar{x}(\cdot)$ is continuous and, thereby, $x_n(\bar{t}_n) \to \bar{x}(\bar{t})$ as $n \to \infty$. Since $\operatorname{dist}(x_n(\bar{t}_n), \mathcal{T}) < 3\alpha_n$ and $\alpha_n \to 0$, then $\bar{x}(\bar{t}) \in \partial\mathcal{T}$ and $u_\natural(\bar{x}(\bar{t})) = 0$ by the definition of a subsolution (it satisfies the boundary condition at $\partial\mathcal{T}$).

Now, we show that $\{u_\natural^{\alpha_n}(x_n(\bar{t}_n))\} \to u_\natural(\bar{x}(\bar{t}))$.

$$\begin{aligned} &\left|u_\natural^{\alpha_n}(x_n(\bar{t}_n)) - u_\natural(\bar{x}(\bar{t}))\right| \\ &= \left|\max_{y \in O_{2\alpha_n}(x_n(\bar{t}_n))} \{u_\natural(y) - w(x_n(\bar{t}_n), y)\} - u_\natural(\bar{x}(\bar{t}))\right| \\ &= \left|u_\natural\Big(y_{\alpha_n}(x_n(\bar{t}_n))\Big) - w\Big(x_n(\bar{t}_n), y_{\alpha_n}(x_n(\bar{t}_n))\Big) - u_\natural(\bar{x}(\bar{t}))\right| \\ &\leq \left|u_\natural\Big(y_{\alpha_n}(x_n(\bar{t}_n))\Big) - u_\natural(\bar{x}(\bar{t}))\right| + \left|w\Big(x_n(\bar{t}_n), y_{\alpha_n}(x_n(\bar{t}_n))\Big)\right|. \end{aligned} \tag{24}$$

Let us consider the first and second terms of the last line of (24) separately. Since we have that $y_{\alpha_n}(x_n(\bar{t}_n)) \in O_{2\alpha_n}(x_n(\bar{t}_n))$, the sequences $x_n(\bar{t}_n) \to \bar{x}(\bar{t})$ and $\alpha_n \to 0$, then $y_{\alpha_n}(x_n(\bar{t}_n)) \to \bar{x}(\bar{t})$. And since $\bar{x}(\bar{t}) \in \partial\mathcal{T}$, and the function $u_\natural(\cdot)$ is continuous on the boundary $\mathcal{T}$ as a subsolution, then the numeric sequence $u_\natural(y_{\alpha_n}(x_n(\bar{t}_n)))$ converges to $u_\natural(\bar{x}(\bar{t}))$ and the first term vanishes as $n \to \infty$.

Now, turn to the second term. Since inequalities

$$\operatorname{dist}(x_n(\bar{t}_n), \mathcal{T}) \leq 3\alpha_n \quad \text{and} \quad \|x_n(\bar{t}_n) - y_{\alpha_n}(x_n(\bar{t}_n))\| \leq 2\alpha_n$$

hold, then $\operatorname{dist}(y_{\alpha_n}(x_n(\bar{t}_n)), \mathcal{T}) \leq 5\alpha_n$. Let

$$\Psi_{\alpha_n} := \max_{z \in \operatorname{cl}(\mathcal{T}_{5\alpha_n} \cap \mathcal{G})} u_\natural(z) =: u_\natural(z_{\alpha_n}). \tag{25}$$

In relation (25), the maximum is achieved, because the function $u_{\natural}(\cdot)$ is upper semicontinuous and the maximum is searched over a compact set. We have a bounded sequence $\{z_{\alpha_n}\}$, where z_{α_n} is defined in (25), from which one can select a subsequence convergent, obviously, to a point at the boundary of the set $\mathcal{T}$ as $n \to \infty$. Let the very sequence $\{z_{\alpha_n}\}$ be convergent. Due to continuity of the function $u_{\natural}(\cdot)$ on the boundary of the set $\mathcal{T}$, we obtain that $\Psi_{\alpha_n} = u_{\natural}(z_{\alpha_n}) \to 0$.

Then,

$$0 \le w_{\alpha_n}\Big(x_n(\bar{t}_n),\, y_{\alpha_n}\big(x_n(\bar{t}_n)\big)\Big) = u_{\natural}\Big(y_{\alpha_n}\big(x_n(\bar{t}_n)\big)\Big) - u_{\natural}^{\alpha_n}\big(x_n(\bar{t}_n)\big) \le \Psi_{\alpha_n} + \alpha_n \to 0.$$

Therefore, both summands in the last line of expression (24) tend to zero as n tends to infinity, so the left-hand side also vanishes. Hence, one gets that $u_{\natural}^{\alpha_n}\big(x_n(\bar{t}_n)\big) \to u_{\natural}\big(\bar{x}(\bar{t})\big)$.

Taking this into account, in (23), we proceed to the limit as $n \to \infty$ (then $\alpha_n \to 0$ and $\delta_n \to 0$) and obtain

$$0 = u_{\natural}\big(\bar{x}(\bar{t})\big) \ge 1 - e^{\bar{t} - \omega_{\natural}} > 0,$$

which is a contradiction. As a result, case (II) is impossible. The theorem is proved. □

5.2 Estimate of Guaranteed Result of the First Player

Now, let us formulate and prove a theorem on the estimate of the guaranteed result of the first player for time-optimal problem with lifeline.

Let $u^{\natural}$ be a supersolution of problem (13), (14). The following transformation of the function $u^{\natural}$ is suggested:

$$u_{\alpha}^{\natural}(x) := \min_{y \in \operatorname{cl} \mathcal{G}} \big\{u(y) + w_{\alpha}(x, y)\big\}.$$

Here, $w_{\alpha}(x, y)$ is defined by equality (16).

We define a feedback strategy $\mathcal{A}_{\alpha} : \operatorname{cl} \mathcal{G} \mapsto A$ of the first player with the equality

$$\mathcal{A}_{\alpha}(x) = a_0\big(x, s^{\alpha}(x)\big). \tag{26}$$

Here, the function a_0 is the extremal pre-strategy identified with the equality

$$a_0(x, s) \in \operatorname{Arg} \min_{a \in A} \Big[\max_{b \in B} \big\langle s, f(x, a, b)\big\rangle\Big],$$

and the vector $s^{\alpha}(x)$ is given by the relation

$$s^{\alpha}(x) := \big(D_x w_{\alpha}\big)\big(x, y^{\alpha}(x)\big) = -\big(D_y w_{\alpha}\big)\big(x, y^{\alpha}(x)\big),$$

where

$$y^{\alpha}(x) \in \operatorname{Arg} \min_{y \in \operatorname{cl} \mathcal{G}} \big[u(y) + w_{\alpha}(x, y)\big].$$

(The indices α are made upper in order to differ from those in definition (17) of the second player's strategy, except for the function w_{α}, which is the same for both situations.)

Theorem 6 *Let Conditions C.1–C.3, and C.4 hold; this guarantees existence and uniqueness of the minimax solution* $u(\cdot) : \operatorname{cl} \mathcal{G} \mapsto \mathbb{R}^n$ *of problem* (13), (14). *Let* x_0 *be a point of the set* $\mathcal{G}$

satisfying the relation $u(x_0) < 1$*. Let* θ *be an arbitrary number such that* $\theta > -\ln(1-u(x_0))$*. Then, there exists a supersolution* $u^\natural$ *of this problem satisfying the relation*

$$\theta > w^\natural := -\ln(1 - u^\natural(x_0)), \tag{27}$$

and numbers $\varepsilon > 0$ *and* $\alpha > 0$*, which meet the estimation*

$$T_1^\varepsilon(x_0, \mathcal{A}_\alpha) \le \theta.$$

Here, $\mathcal{A}_\alpha$ *is a feedback strategy of the first player defined by* (26)*, and the variable* $T_1^\varepsilon(x_0, \mathcal{A}_\alpha)$ *is introduced by equality* (7)*.*

Proof By Proposition 2, we pick out a supersolution $u^\natural$ continuous on the boundary $\mathcal{G}$ in such a way. The minimax solution exists because of the satisfaction of Condition C.4, which results in the existence of the sequence of supersolutions, which are continuous on the boundary of the set $\mathcal{G}$ and converge to this minimax solution. Among the elements of the sequence, one can choose a supersolution so that inequality (27) holds.

We choose some $\varepsilon > 0$ and $\alpha \in (0, \varepsilon/3]$. Also, we choose a sequence of points $\bar{x}_n \to x_0$ and a sequence of partitions $\{\Delta_n\}$ of the positive time semiaxis such that $\delta_n = \operatorname{diam} \Delta_n \to 0$. By these sequences, we obtain a sequence of a sets of constructive motions $\mathbb{X}(\bar{x}_n, \mathcal{A}_\alpha, \Delta_n) =: \mathbb{X}_n^\alpha$.

Let us consider the following cases:

(I) There exist $\bar{\varepsilon} > 0$, $\bar{\alpha} \in (0, \bar{\varepsilon}/3]$ such that for all sequences $\{x_n(\cdot) \in \mathbb{X}_n^\alpha\}$, we obtain that for all $t \le \theta$ the inequality $\operatorname{dist}\big(x_n(t), \mathcal{F}\big) \ge 3\bar{\alpha}$ holds. Herewith, two cases are possible:

(I.1) For every n, there exists a trajectory $x_n(\cdot) \in \mathbb{X}_n^\alpha$, for which there exists an instant $t_n \in \Delta_n$ such that $t_n < \theta$ and $\operatorname{dist}(x_n(t_n), \mathcal{T}) \le 3\bar{\alpha}$.

(I.2) Vice versa, for all n and for every trajectory $x_n(\cdot) \in \mathbb{X}_n^\alpha$, the opposite inequality $\operatorname{dist}(x_n(t), \mathcal{T}) > 3\bar{\alpha}$ holds for all $t \in \Delta_n$ satisfying the restriction $t < \theta$.

(II) Conversely, let for all $\varepsilon > 0$ and $\alpha \in (0, \varepsilon/3]$ there exist a sequence $\{x_n(\cdot) \in \mathbb{X}_n^\alpha\}$ and a collection of instants $\bar{t}_n$, $\bar{t}_n < \theta$, which satisfy the inequality $\operatorname{dist}\big(x_n(\bar{t}_n), \mathcal{F}\big) < 3\alpha$.

Cases (I.1) and (I.2) are considered similarly to cases (i) and (ii) from the proof of Theorem 19.6 of book [16, § 19.6, p. 247]; this proof uses the denotations $\bar{\alpha}$ and $\bar{\varepsilon}$, which are not mentioned further in this text. Let us consider case (II).

Let us recall that

$$u_\alpha^\natural(x) = \min_{y \in \operatorname{cl} \mathcal{G}} \big\{u^\natural(y) + w(x, y)\big\},$$

for all $\tau \in [t_i, t_{i+1}] \cap [0, \theta_0]$, it is true that

$$u_\alpha^\natural\big(x(\tau)\big) \le 1 - \Big[1 - u_\alpha^\natural\big(x(t_i)\big)\Big] e^{\tau - t_i} + (\tau - t_i) e^{\tau - t_i} h(\delta),$$

(it is proved in [16, § 19.6, p. 247–249]), and $u_\alpha^\natural(x_0) \le u^\natural(x_0) + \alpha = 1 - e^{-\omega} + \alpha$.

The instant $\theta_0 = t_{k+1}$ is such that t_k, $t_{k+1} < \theta$, $\operatorname{dist}(x(t_k), \mathcal{F}) > 3\alpha$, and $\operatorname{dist}(x(t_{k+1}), \mathcal{F}) \le 3\alpha$. From these recurrent estimates, one can obtain that

$$u_\alpha^\natural(x(\theta_0)) \le 1 - [1 - u_\alpha^\natural(x_0)] e^{\theta_0} + \theta_0 e^{\theta_0} h(\delta) \le 1 - e^{\theta_0 - \omega^\natural} + e^{\theta_0}\alpha + e^{\theta_0}\theta_0 h(\delta). \tag{28}$$

Similarly to the proof of Theorem 5, we construct convergent sequences

$$\begin{gathered}
\{\varepsilon_n\} \to 0, \quad \big\{\alpha_n : \alpha_n \in (0, \varepsilon_n/3]\big\} \to 0, \\
\{x_n(\cdot) : x_n(\cdot) \in \mathbb{X}_n^{\alpha_n}\} \rightrightarrows \bar{x}(\cdot), \\
\big\{\bar{t}_n : \operatorname{dist}\big(x(\bar{t}_n), \mathcal{F}\big) < 3\alpha_n\big\} \to \bar{t},
\end{gathered}$$

and also $\bar{x}(\bar{t}) \in \partial\mathcal{F}$. By the constructed sequences, we get

$$\left\{u^{\natural}_{\alpha_n}\left(x_n(\bar{t}_n)\right)\right\}, \text{ where } u^{\natural}_{\alpha_n}(x) := \min_{y\in \mathrm{cl}\,\mathcal{G}} \left\{u^{\natural}(y) + w_{\alpha_n}(x, y)\right\}.$$

The proof of the convergence of $\left\{u^{\natural}_{\alpha_n}\left(x_n(\bar{t}_n)\right)\right\} \to u^{\natural}_{\alpha_n}(\bar{x}(\bar{t}))$ entirely repeats the corresponding proof for Theorem 5.

In (28), by choice of the instants t_n, one can substitute θ_0 with t_n. We obtain

$$1 = u^{\natural}\left(\bar{x}(\bar{t})\right) \leq 1 - e^{\bar{t}-\omega^{\natural}} < 1$$

and come to a contradiction. The theorem is proved. □

Thus, it has been justified that the guaranteed result of the first player of problem (1), (10) can be made arbitrary close from above to the minimax solution of problem (13), (14), and the guaranteed result of the second player can be made arbitrary close from below to the minimax solution. Consequently, the guaranteed results coincide with each other and the minimax solution. This results in coincidence of the minimax solution with the value function of game (1), (10).

6 Comments on Classic Time-Optimal Games

As it was said above, in book [16], the proof of Theorem 19.8, which compares the guaranteed result of the second player with the minimax solution of HJE, is omitted. The author says that it is similar to the proof of Theorem 19.6, which compares the guaranteed result of the first player with the minimax solution of HJE. In our attempts to prove Theorem 19.8 similarly to the proof of Theorem 19.6, we come to a conclusion that only estimates on change of the subsolution along the solution of the corresponding characteristic inclusion are analogue; however, a consideration itself is based on additional information on continuity of chosen subsolution on the boundary of the terminal set.

The proof of Theorem 5 can be used for constructing the proof of Theorem 19.8. The proof can be broken apart in two cases: if there exists a second player's strategy, which up to the instant θ repels the motion of the system from some neighborhood of the terminal set, or if there is no such a strategy. The first case is obvious, and the second is proved in the same way as case (II) in Theorem 5.

It is worthwhile to say that, in general case, Theorem 19.8 cannot be obtained as a consequence of Theorem 5 when $\mathcal{G} \to \mathbb{R}^n$ because of the strong assumptions on the boundary $\partial\mathcal{G}$ of the set $\mathcal{G}$ (smoothness and capability of the second player to push the system through the boundary $\partial\mathcal{F}$). However, in some particular situations, if one successfully constructs a sequence of growing sets $\mathcal{G}$ satisfying the assumptions, then Theorem 19.8 can be proved via Theorem 5.

7 Conclusion

This paper justifies existence of the value function of time-optimal problems with lifeline. Existence of a minimax solution of the boundary value problem corresponding to a time-optimal differential game with lifeline is proved under sufficiently strong assumptions about the boundary of the set, where the game takes place. Under the same assumptions, coincidence of the minimax solution with the value function of a time-optimal game with lifeline is shown.

Assumptions made in the paper are that the boundary of the terminal set and the lifeline are smooth manifolds. Moreover, it is assumed that the boundary of the terminal set is an admissible zone for the first player (that is, if the system is at the boundary of the terminal set, then the first player guarantees leading the trajectory of the system inside the terminal set), and the lifeline is an admissible zone for the second player (that is, if the system is at the lifeline, then the second player guarantees leading the trajectory of the system outside the set where the game takes place). These assumptions together guarantee continuity of the value function. It is planned to weaken the assumptions, in terms of which the theorems are proved, and to find a connection between the value function of the classical time-optimal problem with the value function of the time-optimal problem with the lifeline.

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