On Some Sufficient Conditions of Existence of a Majority Committee

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Abstract—The questions of existence of the majority committee for the nonlinear systems of inclusions in $R^n$ are discussed. The conditions sufficient for the arbitrary inclusions in $R^n$ to be solved by the majority committee are indicated. In particular, these conditions are presented for the systems of polyhedral sets in a plane.

INTRODUCTION

The construction of a decision rule in pattern recognition is related, first of all, to the necessity of choosing an element from the set described by the system of restrictions. In a number of cases, this set turns out to be empty, i.e., the system is inconsistent; it often happens in the problems of technical and economic diagnosis as well as in the problems of medical diagnostics. However, we can fit the substitution for the solution of this problem. A committee theory studies the ways of solving inconsistent systems.

Suppose we have a system

$$x \in D_j, j \in N_m \left( \bigcap_{j \in N_m} D_j = \emptyset \right) \tag{1}$$

Definition 1. We call a finite sequence $Q = (q_1, q_2, \ldots, q_k)$ a committee (a majority committee) of system (1) if for any $j \in N_m$

$$|\{i \in N_k : q_i \in D_j\}| > \frac{k}{2}$$

holds [1].

At present, the committees of the systems of linear inequalities are investigated quite well—existence theorems are proven and the number of members for the minimal (according to the number of elements in the sequence) committee is estimated [2, 5]. To obtain such a result for the systems with different types of constraints seems to be interesting. This paper concerns some sufficient conditions of existence of the majority committee for systems of inclusions in $R^n$.

**SYSTEMS OF INCLUSIONS IN $R^n$**

Let $D_j \subset R^n$ for any $j \in N_m$ in system (1).

Definition 2. A maximum, according to the inclusion, set $K(C, x) \subset R^n$ is called a recession cone of the set $C$ at the point $x \in C$ if the inclusion $x + \lambda y \in C$ is valid for any $y \in K(C, x)$ and $\lambda \geq 0$.

Thus defined set $K(C, x)$ may not contain all the directions of unboundedness of the set $C$. If the set $C$ is convex, the recession cone $K(C, x)$ coincides with the recession cone $0^*C$ of the set $C$ for any $x \in C$ [4].

Theorem 1. Let $K(C, x) \subset R^n$ be a recession cone of the set $C \subset R^n$ at the point $x$. Then, for any $y \in \text{int}K(C, x)$ there is $\lambda > 0$, such that for any $\tilde{x} \geq \lambda$, $\tilde{x}y \in C$.

Proof. Let $y \in \text{int}K(C, x)$. Then, there is a set of affinely independent vectors $y_1, y_2, \ldots, y_{n+1}$, such that

$$y = \sum_{i=1}^{n+1} \alpha_i y_i, \quad \alpha_i > 0, \quad \sum_{i=1}^{n+1} \alpha_i = 1, \quad \text{and} \quad \text{conv}\{y_1, y_2, \ldots, y_{n+1}\} \subset \text{int}K(C, x).$$

Definition 2 implies that for any $\tilde{\lambda} > 0$ if $i \in N_{n+1}$, then $x + \tilde{\lambda} y_i \in C$.

Consider the $(n + 1)$st sequence of the form $\lambda_k x_{ik}$, $i \in N_{n+1}$ where $\lambda_k = \frac{1}{k}$ and $x_{ik} = x + ky_i$. By construction, $\lim_{k \to \infty} \lambda_k x_{ik} = y$, hence, there is $k$ such that, on the one hand, the set of vectors $\{x_{1k}, x_{2k}, \ldots, x_{n+1,k}\} \subset C$ turns out to be affinely independent and, on the other hand, the set of vectors $\{\lambda_k x_{1k}, \lambda_k x_{2k}, \ldots, \lambda_k x_{n+1,k}\}$ is also affinely independent. In addition, $y = \sum_{i=1}^{n+1} \gamma_i \lambda_k x_{ik} = \frac{\sum_{i=1}^{n+1} \gamma_i x_{ik}}{k}$ and for all $i \in N_{n+1}$: $\gamma_i > 0$, $\sum_{i=1}^{n+1} \gamma_i = 1$. Let us

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introduce $\lambda = \bar{\lambda}$. Then,
\[
\lambda y = \sum_{i=1}^{n+1} \gamma_i y_i = \sum_{i=1}^{n+1} \gamma_i (x + \bar{\lambda} y_i) = x + \bar{\lambda} \sum_{i=1}^{n+1} y_i =
\]
due to the fact that for any $\delta \geq 0: x = \delta \text{conv}\{y_1, y_2, \ldots, y_{n+1}\} \subseteq C$. Therefore, we found $\lambda$ which satisfies inclusion $\lambda y \in C$.

Let us now demonstrate that $\bar{\lambda} y \in C$ for all $\bar{\lambda} \geq \lambda$.

Let $\bar{\lambda} > \lambda$ and $\lambda = \bar{\lambda} - \lambda$. Consider $\lambda y = \lambda y + \bar{\lambda} y$. From the above reasoning, we know that $\lambda y = x + \bar{\lambda} \sum_{i=1}^{n+1} y_i$ and $y = \sum_{i=1}^{n+1} \alpha_i y_i$, whence $\bar{\lambda} y = x + \bar{\lambda} \sum_{i=1}^{n+1} y_i + \lambda \sum_{i=1}^{n+1} \alpha_i y_i = x + \sum_{i=1}^{n+1} \left(\bar{\lambda} y_i + \lambda \alpha_i y_i\right)$. Since
\[
\left(\sum_{i=1}^{n+1} (\bar{\lambda} y_i + \lambda \alpha_i)\right) \subseteq \text{conv}\{y_1, y_2, \ldots, y_{n+1}\},
\]
then $\bar{\lambda} y \in C$. The theorem is proven.

**Corollary 1.** Let $K(C, x) \subset R^n$ be a recession cone of the set $C \subset R^n$ at the point $x$. Then, for any $y \in \text{int}K(C, x)$ and $z \in R^n$ there is $\lambda > 0$, such that $z + \lambda y \in C$ for all $\lambda \geq \lambda$.

**Proof.** Let $z \in R^n$ and $y \in \text{int}K(C, x)$. Let us denote
\[
T = \{x + \delta \text{conv}\{y_1, y_2, \ldots, y_{n+1}\}, \delta \geq 0\},
\]
where vectors $y_1, y_2, \ldots, y_{n+1}$ are selected in the same way as in the proof of Theorem 1. There is $\mu > 0$ for which inclusion
\[
y + \mu z \in \text{conv}\{y_1, y_2, \ldots, y_{n+1}\}
\]
is valid.

We can choose $\lambda > 0$ such that $\lambda y \in T, \bar{\lambda} (y + \mu z) \in T$, and $\lambda + \bar{\lambda} \mu \geq 1$ for any $\bar{\lambda} \geq \lambda$. Since $T$ is a convex set, then $z + \bar{\lambda} y \in T \subset C$ for any $\bar{\lambda} \geq \lambda$. This means $z = \bar{\lambda} y \in C$, Q.E.D.

Along with system (1), let us consider the following system:
\[
x \in \text{int} K(D_j, u_j), \quad j \in N_m,
\]
where $K(D_j, u)$ is a recession cone of the set $D_j$ at the point $u_j \in D_j (j \in N_m)$.

**Theorem 2.** Existence of the committee of system (1) is provided by the existence of the committee of the system (2) for some $u_j \in D_j (j \in N_m)$.

**Proof.** Let $Q = (q_1, q_2, \ldots, q_3)$ be the committee of system (2). Let us denote $V_j = \{i \in N_k : q_i \in \text{int}K(D_j, u_j)\}, j \in N_m$. By Definition 1, $|V_j| > \frac{k}{2}, j \in N_m$. According to Theorem 1, there are $\lambda_{ji} > 0, j \in N_m, i \in N_{V_j}$, such that $\lambda_{ji} q_i \in D_j, j \in N_m, i \in N_{V_j}$. Denote also $\lambda = \max_{j \in N_m, i \in N_{V_j}} \lambda_{ji}$ and $Q' = (\lambda q_1, \lambda q_2, \ldots, \lambda q_k) = (q_1, q_2, \ldots, q_k)$. By construction,
\[
|\{i \in N_k : q_i \in D_j\}| \geq |V_j| > \frac{k}{2},
\]
holds for any $j \in N_m$ and, therefore, the sequence $Q$ is the committee of system (1).

One should notice that if the committee of system (2) exists, the number of members of minimal committee of system (1) is limited to the number of members of minimal committee of system (2).

Let us demonstrate that existence of the committee of system (2), generally, is not a necessary condition for existence of the committee of system (1). Consider the following system of inclusions over $R^2$:
\[
\begin{align*}
x & \in D_1 \Rightarrow \{x \in R^2 : -3x^1 - 2x^2 + 6 \geq 0, x^1 + 2x^2 - 2 \geq 0\} \\
x & \in D_2 \Rightarrow \{x \in R^2 : 2x^1 - 3x^2 + 6 \geq 0, -2x^1 + 2x^2 - 2 \geq 0\} \\
x & \in D_3 \Rightarrow \{x \in R^2 : x^1 - 2x^2 \geq 0, -x^1 + 4x^2 + 4 \geq 0\}.
\end{align*}
\]

A sequence
\[
Q = (x_1 = (2, 0), x_2 = (0, 2), x_3 = (-2, -3))
\]
is an example of the committee of this system.

However, the system
\[
\begin{align*}
x & \in \text{int} K(D_1) \Rightarrow \{x \in R^2 : -3x^1 - 2x^2 + 6 > 0, x^1 + 2x^2 - 2 > 0\} \\
x & \in \text{int} K(D_2) \Rightarrow \{x \in R^2 : 2x^1 - 3x^2 + 6 > 0, -2x^1 + 2x^2 - 2 > 0\} \\
x & \in \text{int} K(D_3) \Rightarrow \{x \in R^2 : x^1 - 2x^2 > 0, -x^1 + 4x^2 + 4 > 0\},
\end{align*}
\]

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has no committee because it consists of mutually disjoint sets and, thus, does not meet the necessary conditions for existence of a committee.

Suppose now that \( L(K(D_j, u_j)) \) is a linear hull of the recession cone \( K(D_j, u_j) \) of the set \( D_j \) at the point \( u_j \in D_j, j \in N_m \). According to Definition 2, inclusion \( u_j + K(D_j, u_j) \subseteq D_j \) holds for any \( u_j \in D_j, u_j \in D_m \). It is quite possible that the following condition

\[
\text{for all } j \in N_m, u_j \in L(K(D_j, u_j))
\]

is fulfilled for a certain set \( u_j \in D_j, j \in N_m \).

We use this condition to formulate a refined version of Theorem 2:

**Theorem 3.** Let condition (3) be valid for some \( u_j \in D_j, j \in N_m \). Then, existence of a committee for the system

\[
x \in \mathbb{R}^n : x = \sum_{i=1}^{t_j} \delta_i p_{ji}, \delta_i \geq 0
\]

is a sufficient condition for existence of a committee for system (1).

**Proof.** Suppose that \( Q = (q_1, q_2, \ldots, q_k) \) is a committee of system (4). Since the linear hull \( L(K(D_j, u_j)) \) of the set \( K(D_j, u_j) \) coincides with its affine hull, then for all \( j \in N_m, r_i K(D_j, u_j) \subseteq L(K(D_j, u_j)) \). Condition (3) is considered fulfilled, therefore, \( u_j + r_i K(D_j, u_j) \subseteq D_j \cap L(K(D_j, u_j)) \). Indeed, let \( y \in r_i K(D_j, u_j) \), then \( y \in L(K(D_j, u_j)) \). On the other hand, \( u_j + r_i K(D_j, u_j) \) and for any \( \lambda \geq 0, u_j + \lambda y \in D_j \), but \( u_j + \lambda y \in (K(D_j, u_j)) \), whence \( u_j + \lambda y \in D_j \cap L(K(D_j, u_j)) \). According to Theorem 1, for any \( y \in r_i K(D_j, u_j) \) and \( z \in L(K(D_j, u_j)) \) there is \( \lambda > 0 \) such that for all \( \lambda > \lambda \) the inclusion \( z + \lambda y \in D_j \cap L(K(D_j, u_j)) \) is valid. Then, we can find \( \lambda > 0 \) such that for sequence \( Q' = (q_1, q_2, \ldots, q_k) = (\lambda q_1, \lambda q_2, \ldots, \lambda q_k) \) the inequality \( \{ i \in N_k : q_i \in D_j \cap L(K(D_j, u_j)) \} \geq k \) holds for all \( j \in N_m \). The theorem is proven.

Theorems 2 and 3 imply a theorem about the sufficient conditions for existence of a committee for the system of the polyhedral sets in \( \mathbb{R}^n \). Let us consider the system of inclusions

\[
x \in S_j, \quad j \in N_m,
\]

where \( S_j = \{ x \in \mathbb{R}^n : a_j x \geq b_j \} \). According to [3], each polyhedral set \( S_j \in N_m \) may be represented in the form \( S_j = P_j^\delta + Q_j^\delta \), where

\[
P_j^\delta = \{ x \in \mathbb{R}^n : x = \sum_{i=1}^{t_j} \gamma_{ji} p_{ji}, \gamma_{ji} \geq 0, \sum_{i=1}^{t_j} \gamma_{ji} = 1 \}
\]

and

\[
Q_j^\delta = \{ x \in \mathbb{R}^n : x = \sum_{i=1}^{t_j} \delta_i p_{ji}, \delta_i \geq 0 \}.
\]

Here, \( P_j^\delta \) is a convex closed bounded polyhedron and \( Q_j^\delta \) is a convex closed cone (\( j \in N_m \)). For each \( j \in N_m \) let us introduce a set:

\[
Q_j^\delta = \{ x \in \mathbb{R}^n : x = \sum_{i=1}^{t_j} \delta_i p_{ji}, \delta_i > 0 \} = \text{int} Q_j^\delta.
\]

**Theorem 4.** Existence of a committee for the system of inclusions

\[
x \in Q_j^\delta, \quad j \in N_m
\]

is a sufficient condition for existence of a committee for system (5).

Let us now consider a system of linear inequalities

\[
(a_i, x) \geq d_i, \quad i \in N_r,
\]

composed of a restrictions to system (5). In other words, \( N_r = \bigcup I_j \), therefore, \( S_j = \{ x \in \mathbb{R}^n : (a_i, x) \geq d_i, i \in I_j \} \). The problem of the committee construction for system (5) may be interpreted as a problem of the committee construction for system (6) \( Q = (x_1, x_2, \ldots, x_k) \) with an additional requirement on its members.

Let \( V_i = \{ i \in N_k : (a_i, x_i) \geq d_i \} \) be a set of numbers of the committee members, which “vote” for \( i \)th inequality. Then, the above requirement may be formalized as follows:

\[
\text{for all } j \in N_m \Rightarrow \left| \bigcap_{i \in I_j} V_i \right| > k/2.
\]

In other words, in order that a majority committee of system (6) is the same for system (5), it is necessary and sufficient that (7) is fulfilled. Indeed, if the committee \( Q = (x_1, x_2, \ldots, x_k) \) for system (6), which is composed of the inequalities of system (5), possesses property (7), then \( \left| \bigcap_{i \in N_k} A_i x_i \geq b_i \right| > k/2 \) is valid, i.e., \( Q \) is a committee of system (5). However, if a committee for system (6) is a committee for system (5), then, for each \( j \in N_m \) the inequality

\[
\left| \bigcap_{i \in I_j} V_i \right| > k/2
\]

necessarily holds.

Theorems 2—4 allow us to substitute a question about existence of a committee for a system of inclusions in \( \mathbb{R}^n \) for a question about existence of a committee of another specific system (it may coincide with the initial system). Generally, this approach yields the problem of the same complexity. Further we try to show some cases,
where existence of a committee for (2)-type systems can be ensured by relatively simple conditions.

Let us consider system (1) over $R^2$. Suppose that some $u_i \in D_i$ are selected. Denote for short $K_j = K(D_i, u_j)$ ($j \in N_m$). Let us now introduce a condition (*): we will think that for all $j \in N_m$ at least one of the cones $K_j$, $R^2/K_j$, is convex and for any $i \neq j \in N_m \cap int K_i \subset int K_j$ is fulfilled.

Theorem 5. If the condition (*) is fulfilled for system (2) over $R^2$, and for any $i, j \in N_m$ the cones $int K_i$ and $int K_j$ intersect in pairs, then there exists a committee for this system and it consists of no more than $m$ members. Proof. For each $j \in N_m$ consider a set $bd K_j = cl K_j \cap int K_j$.

By virtue of condition (*), it can be represented as

$$bd K_j = \{ \mu z_{j1} : \mu \geq 0 \} \cup \{ v z_{j2} : v \geq 0 \},$$

where $z_{j1}, z_{j2}$ are some vectors from $cl K_j$. Then, for any $i \neq j$ either $z_{j1} \in int K_i$ or $z_{j2} \in int K_j$ is valid. Let us introduce some auxiliary notation:

$$I_{j1} = \{ i \in N_m : i \neq j, z_{j1} \in int K_i \},$$

$$I_{j2} = \{ i \in N_m : i \neq j, z_{j2} \in int K_i \},$$

$$S(J) = \{ x \in R^2 : x \in \bigcap_{j \in J} int K_j, J \subseteq N_m \}.$$

Thus, for each $j \in N_m$ it follows:

$$I_{j1} \cup I_{j2} \cup \{ j \} = N_m. \quad (8)$$

Consider the case when $j = 1$. For this number, $|I_{11} \cup I_{12} \cup \{ 1 \}| = N_m$ and, therefore, $|I_{11}| + |I_{12}| + 1 \geq m$. Let, for example, $|I_{11}| \geq |I_{12}|$. Then, $2|I_{11}| + 1 \geq m$, whence $|I_{11}| + 1 \geq m + 1/2$.

Let us construct the following sequence:

$$q_{00} \in S(I_{11} \cup \{ 1 \}),$$

$$q_{11} \in S(I_{11} \cup \{ j \}),$$

$$q_{j2} \in S(I_{j2} \cup \{ j \}),$$

$$j \in N_m/(I_{11} \cup \{ 1 \}).$$

Note, that $|N_m/(I_{11} \cup \{ 1 \})| = m - |I_{11}| + 1 < m - m + 1/2 = m - 1/2$. Therefore, the sequence consists of no more than $1 + 2(m - 1)/2 = m$ members. Let us demonstrate that $Q = (q_{00}, q_{11}, q_{12}, q_{23}, q_{24}, \ldots, q_{k1}, q_{k2})$ where $k = \left\lfloor \frac{m - 1}{2} \right\rfloor$ is a committee of system (2). By virtue of (8) for any $l \in N_m$ we get

$$p = \left\{ q_{j1} : q_{j1} \in int K_j \right\} + \left\{ q_{j2} : q_{j2} \in int K_j \right\} \geq \begin{cases} k + 1, & \text{if } l \in N_m/(I_{11} \cup \{ 1 \}); \\ k, & \text{if } l \in I_{11} \cup \{ 1 \}. \end{cases}$$

This implies $p + \{|j \in N_k : q_{00} \in int K_j\} \geq k + 1$. It means that the constructed sequence $Q$ really is a majority committee for system (2) and consists of no more than $m$ members. Theorem 5 indicates sufficient conditions for the existence of a committee for the system of cones (not necessarily convex) over $R^2$. If in addition to the condition (*), we require the fulfillment of the equality $int K_j = K_j$ for any $j \in N_m$, (i.e., for any $j \in N_m K_j$ is an open set), then nonempty intersection of each pair $K_i$ and $K_j$ is not only a sufficient, but also a necessary condition for existence of a committee. Here, the necessity follows from the fact that the pairwise intersection of sets is the necessary condition for existence of a committee for an arbitrary system of inclusions (1).

MAJORITY COMMITTEE FOR A SYSTEM OF POLYHEDRAL SETS IN $R^2$

Suppose an inconsistent system

$$(a_j, x) > 0, \quad j \in N_m \quad (9)$$

is set over $R^2$. Suppose also that there are no zero and opposite vectors among $a_i$. We call the consistent subsystem of system (9) with the index $J \subseteq N_m$ such that any subsystem with the index $J \subseteq (J \neq J)$ is inconsistent, maximum consistent subsystem (MCS) of system (9).

A set of indices of maximum consistent subsystems of system (9) we denote as $J_1, J_2, \ldots, J_k$. Let $Q = (q_1, q_2, \ldots, q_k)$ be the committee of the system composed of MCS solutions.

Lemma 1. For any $i \in N_k$, there exists such $l \in N_i$ that $q_i$ is a solution of a subsystem of system (9) with the index $J_i$.

Proof. Towards the contradiction, suppose that MCS with index $J^* = J_1$ has no solution among the committee $Q$ members. Let the cone of solution of MCS with the index $J^*$ be determined by the pair of vectors $a_{j^1}, a_{j^2}$, where $j^1, j^2 \in N_m$. Let us introduce the following notation:

$$A = \left\{ x \in R^2 : (a_{j^1}, x) > 0, (a_{j^2}, x) > 0 \right\},$$

$$B = \left\{ x \in R^2 : (a_{j^1}, x) > 0, (a_{j^2}, x) \leq 0 \right\},$$

$$C = \left\{ x \in R^2 : (a_{j^1}, x) > 0 \right\},$$

$$D = \left\{ x \in R^2 : (a_{j^2}, x) \leq 0 \right\}.$$
By the assumption, none of the committee \( Q \) members belongs to the set \( A \). Then, more than a half of them lie in the set \( B \). Since \( B \subset D \), then the set \( C \) contains less than a half of them and, thus, less than a half of the committee \( Q \) members vote for the inequality \( j \). Hence, \( Q \) is not a committee of system (9). The lemma is proven.

**Corollary 2.** A committee of system (9) composed of the solutions of all of the MCSs taken one by one is minimal.

**Proof.** According to the conditions imposed on system (9), the committee \( Q \) of the solutions of all MCSs taken one by one exists. Since any committee must contain at least one solution of each MCS, then committee \( Q \) is minimal.

Now consider the system

\[
x \in K_j, \quad j \in N_m,
\]

so that \( K_j \ (j \in N_m) \) are open convex intersecting in pairs cones in \( R^2 \). According to Theorem 5, there is a committee \( Q = (q_1, q_2, \ldots, q_k) \) of system (10), where \( k \leq m \).

We consider the solutions of some MCSs of system (10) as members of a committee. Let us put system (10) into correspondence with the system of strict homogeneous linear inequalities

\[
(a_{1j}, x) > 0, \quad (a_{2j}, x) > 0 \quad j \in N_m,
\]

so that \( K_j = \{ x \in R^2 : (a_{2j}, x) > 0 \} \) for all \( j \in N_m \). Sets of solutions of MCSs of system (10) are the sets of solutions of some MCSs of system (11). Since the committee \( Q \) of system (10) is also a committee of system (11) and by virtue of Lemma 1, for each MCS of system (11) there is such a member in the committee \( Q \) that is the solution of this MCS. Thus, the set of solutions of some MCS of system (10) is the set of solutions of some MCS of system (11) and vice versa. Now let us demonstrate that for any \( j \in N_m \), a pair of inequalities \( (a_{1j}, x) > 0 \) and \( (a_{2j}, x) > 0 \) is a member of more than a half of MCSs of system (11).

Without loss of generality we will consider \( a_{1j} \parallel a_{2j}, \) i.e., cone \( K_j \) is not a half-plane. For this pair of inequalities, let us find inequality \((\hat{a}, x) > 0\) in the system such that

\[
K_j \subset \{ x \in R^2 : (\hat{a}, x) > 0 \},
\]

and include it in the system. We can find this inequality due to convexity of the cone \( K_j \) and noncollinearity of the vectors \( a_{1j} \) and \( a_{2j} \). Thus, we obtain the system:

\[
(a_{1j}, x) > 0, \quad (a_{2j}, x) > 0 \quad j \in N_m,
\]

\[
(\hat{a}, x) > 0.
\]

The following lemma is valid:

**Lemma 2.** Systems (11) and (13) contain an equal number of MCSs and their solution sets coincide.

**Proof.** On adding further constraints to the system, the number of MCSs does not decrease. Suppose the number of MCSs increases as a result of transition from system (11) to system (13). This means, that there is a cone \( K_j \) such that the solution set of a new MCS coincides with the set \( K_j \cap \{ x \in R^2 : (\hat{a}, x) > 0 \} \). Since (12) is valid and according to the conditions imposed on system (10), \( K_j \cap K_{j'} \neq \emptyset \), then

\[
(K_j \cap K_{j'}) \subset \{ x \in R^2 : (\hat{a}, x) > 0 \}.
\]

Therefore, \( K_j \cap \{ x \in R^2 : (\hat{a}, x) > 0 \} \) is not a solution set of MCS and our assumption that the number of MCSs changes as a result of transition from one system to another is wrong. On the other hand, due to (14), solution sets of the MCS coincide in systems (11) and (13). The lemma is proven.

Owing to the pairwise cone intersection of system (10), system (13) does not contain opposite vectors. Therefore, we can construct its committee \( C = (c_1, c_2, \ldots, c_t) \) from the solutions of all MCSs taken one by one. By the definition of a committee, the set of solutions of inequality \((\hat{a}, x) > 0\) must contain more than a half of the members of committee \( C \), that is,

\[
(i \in N_t : (\hat{a}, c_i) > 0) > \frac{t}{2}.
\]

Hence, the inequality \((\hat{a}, x) > 0\) belongs to more than a half of MCSs of system (13) and, therefore, due to the fact that the solution sets of MCSs coincide in systems (10) and (11), and according to Lemma 2, the cone \( K_j \) is a part of more than a half of MCSs of system (10).

The above reasoning yields the following theorem.

**Theorem 6.** Suppose system (9) is set over \( R^2 \) and there are no zero or opposite vectors among vectors \( a_j (j \in N_m) \). Then, for any partition \( N_m = I_1 \cup I_2 \cup \ldots \cup I_{m'} \) such that the sets of solutions of the subsystems with indices \( I_l (l \in N_p) \) intersect by pairs, the totality of the solutions of MCSs of system (9) taken one by one forms a committee of the system

\[
x \in K_l = \{ x \in R^2 : (a_j, x) > 0, j \in I_l \}, \quad l \in N_p
\]

and, moreover, it is a minimal committee.

**Proof.** The conclusion of the theorem is valid because systems (15) and (9) have the same set of MCSs. In addition, for any index \( I_l (l \in N_p) \), the sets of solutions of no less than a half of MCSs of system (9) belong to the solution set of a correspondent subsystem. The committee mentioned in the conclusion of the theorem is minimal by virtue of Corollary 2.

Now, let us turn to the systems of polyhedral sets over \( R^2 \). Let

\[
x \in S_l, \quad l \in N_r.
\]

be the system of open polyhedral sets \((S_l \subset R^2, \quad l \in N_r)\). Consider a system of the strict linear nonhomogeneous inequalities which compose polyhedral sets of system (16):

\[
(a_j, x) > b_j, \quad j \in N_m = \bigcup_{l \in N_r} I_l
\]
so that \( S_l = \{ x \in \mathbb{R}^2 : (a_j, x) > b_j, j \in I_l \} \). Let
\[
(a_j, x) > 0, \quad j \in N_m
\] (17)
be the corresponding homogeneous system.

The polyhedral set \( S_l \) may be represented in the form \( S_l = P_{j_1}^l + Q_{j_2}^l \). Suppose that cones \( Q_{j_2}^l (l \in N_l) \) intersect in pairs. According to Theorem 5, there is a majority committee \( C \) of system (16). Consider the subsystem of system (17), which has the index \( I_l \). The inequalities of this subsystem correspond to the inequalities of system (16), which specify the \( l \)th polyhedral set. Suppose that the inequalities with the numbers \( j_1^l, j_2^l \in I_l \) define the faces of the solution cone of the subsystem with the index \( I_l \). Let us demonstrate that any inequality with the number \( j \in I_l \backslash \{ j_1^l, j_2^l \} \) follows from the system \( x \in Q_{j_1}^l = \{(a_j, x) > 0, (a_j, x) > 0 \} \). Let \( x \) be a solution of this system, i.e., \( (a_j, x') > 0 \) and \( (a_j, x') > 0 \). Suppose we find \( j \in I_l \backslash \{ j_1^l, j_2^l \} \) such that \( (a_j, x') < 0 \) (if it equals zero, we may correct \( x \) insignificantly).

According to Theorem 1, there is \( \lambda \geq 0 \) such that for any \( \lambda' > \lambda \) and \( j \) the inequality \( (a_j, \lambda' x') > b_j \) is fulfilled. Meanwhile, we can find such \( \lambda^* > \lambda \) that \( (a_j, \lambda^* x') > b_j \), which contradicts our supposition. Therefore, any inequality of the subsystem with the index \( I_l \), different from inequalities \( j_1^l, j_2^l \), which define the faces of the solution cone, follows from the system of these inequalities. According to Lemma 2, these inequalities do not affect the number of members of the solution cone, and the subsystem \( I_l \) is a majority committee for an inconsistent system of arbitrary inequalities.

According to Theorem 7, suppose that \( S_l = P_{j_1}^l + Q_{j_2}^l \) and the cones \( Q_{j_2}^l, l \in N_l \) intersect in pairs in the system of polyhedral sets (16) over \( \mathbb{R}^2 \). Then, for any committee \( \lambda = (c_1, c_2, ..., c_k), (k \leq m), \) of system (17), which consists of the solutions of its MCSs taken one by one, there is \( \lambda \geq 0 \) such that for all \( \lambda' > \lambda \) the sequence \( C = (\lambda c_1, \lambda c_2, ..., \lambda c_k) \) is a majority committee of system (16).

The above reasoning and Theorem 7 set up correspondence between the committees of systems of polyhedral sets in \( \mathbb{R}^2 \) and committees of systems of strict homogeneous linear inequalities. In particular, under certain assumptions concerning the system of polyhedral sets (16), the construction of its committee \( C \) is reduced to the construction of the corresponding system of linear homogeneous inequalities (17). Here, the length of the equation \( C \) does not exceed \( r \), the number of inclusions in system (16).