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APPROXIMATE SOLUTIONS OF DIFFERENTIAL GAMES USING MIXED STRATEGIES

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This paper is concerned with the numerical solution of differential games using mixed strategies. Mixed strategies are defined [1] as functions which associate a probability measure with each position of the game. In a discrete control scheme these strategies could be realized by means of stochastic approximation procedures.

Let a conflict-controlled system be described on the interval $[t_*, \vartheta]$ by the equation

$$\frac{dx}{dt} = f(t, x, u, v), \ x \in R^n, \ u \in P, \ v \in Q \quad . \tag{1}$$

Here x is the phase vector, u is the control parameter of the first player, v is the control parameter of the second one, and P and Q are compact sets in R^p and R^q, respectively. The function f is continuous with respect to all its variables and Lipschitzian in x. We also assume that the function f satisfies a condition concerning the extension of solutions.

Let M be a compact set in R^n. The aim of the first player is to direct the system (1) into set M at time ϑ. The aim of the second player is to prevent this from happening. The mixed strategies of the first and second players are the functions which associate probability measures $\mu(t, x)$ on P and $\nu(t, x)$ on Q with each position (t, x).

It is known [1] that for any initial position (t_0, x_0) there exists either a mixed strategy for the first player which solves the approach problem, or a mixed strategy for the second player which solves the evasion problem. Therefore it is important to construct the set W^0 of all initial positions from which the approach problem can be solved.

According to [1,2], the set W^0 is the maximal stable bridge and can be determined as the maximal closed set in the space of positions (t, x) which satisfies the conditions:

1. $W_{\vartheta}^{0} \subset M$;

2. $W_{\tau_*}^{0} \subset \pi(\tau_* \,;\, \tau^* \,,\, W_{\tau^*}^{0})$ for all τ_*, τ^* such that $t_* \le \tau_* < \tau^* \le \vartheta$.

Here

$$W_{\tau}^{0} = \{x \in R^{n} : (\tau , x) \in W^{0}\} \quad ,$$

$$\pi(\tau_* \,;\, \tau^* , W_{\tau^*}^{0}) = \bigcap_{l \in S} \mathbf{X}_l(\tau_* \,;\, \tau^* , W_{\tau^*}^{0}) \quad ,$$

$$S = \{l \in R^{n} : \|l\| = 1\} \quad ,$$

$$X_l(\tau_* \,;\, \tau^* , W_{\tau^*}^{0}) = \{x \in R^{n} : \mathbf{X}_l(\tau^* \,;\, \tau_* , x) \cap W_{\tau^*}^{0} \neq \phi\} \quad .$$

$\mathbf{X}_l(\tau^* \,;\, \tau_* , x_*)$ denotes the set of all points $y \in R^{n}$ for which there exists a solution $x(t)$ $(t \in [\tau_* , \tau^*],\ x(\tau_*) = x_*)$ of the differential inclusion

$$\dot{x} \in F_l(t , x) \, , \; F_l(t , x) = G_l(t , x) \cap O \tag{2}$$

$$G_l(t , x) = \{g \in R^{n} : l'g \le \max_{\mu \in \{\mu\}} \min_{\nu \in \{\nu\}} \int_P \int_Q l'f(t , x , u , v)\mu(\mathrm{d}u)\nu(\mathrm{d}v)\}$$

such that $x(\tau^*) = y$. Here $\{\mu\}$ and $\{\nu\}$ are sets of probability measures on P and Q, and O is a closed ball of sufficiently large radius.

Using the results of [2] we can establish that W^{0} is the limit of the systems of sets $\{W_{t_i} : t_i \in \Gamma_m\}$, where Γ_m, $m = 1,2, \ldots,$ is a subdivision of the interval $[t_* , \vartheta]$ and its diameter $\Delta(\Gamma_m)$ approaches zero as $m \to \infty$.

For every subdivision $\Gamma_m = \{t_1 = t_* , t_2 , \ldots , t_{N(m)} = \vartheta\}$, the system $\{W_{t_i} : t_i \in \Gamma_m\}$ is defined by the recursive relations

$$W_{t_i} = \bar{\pi}(t_i \,;\, t_{i+1} , W_{t_{i+1}}) \, , \; i = 1,2 , \ldots , N(m) - 1$$

$$W_{t_{N(m)}} = M_{\varepsilon_{N(m)}} \quad ,$$

where

$$\bar{\pi}(t_i \,;\, t_{i+1} , W_{t_{i+1}}) = \bigcap_{l \in S} X_l(t_i \,;\, t_{i+1} , W_{t_{i+1}}) \tag{3}$$

$$X_l(t_i \,;\, t_{i+1} , W_{t_{i+1}}) = \{x \in R^{n} : X_l(t_{i+1} \,;\, t_i , x) \cap W_{t_{i+1}} \neq \phi\}$$

$$X_l(t_{i+1} \,;\, t_i , x) = x + (t_{i+1} - t_i)F_l(t_i , x) \;;$$

$\varepsilon_{N(m)}$ denotes some non-negative number, and M_{α} is a closed α-neighborhood of the set M.

These relations can be used as a basis for algorithms designed to compute the set W^0. We shall discuss a numerical procedure for system (1) with $f(t, x, u, v) = A(t)x + \varphi(t, u, v)$.

Let the dynamics of the system be described by the quasi-linear differential equation

$$\dot{x} = A(t)x + \varphi(t, u, v), \ x \in R^n, \ u \in P, \ v \in Q \quad . \tag{4}$$

Assume that the terminal set $\mathbf{M}$ depends on only k coordinates, i.e., $\mathbf{M} = \{x \in R^n : (x)_k \in M\}$, where $(x)_k$ is the vector of chosen coordinates. Let $\mathbf{M}$ be a closed, convex and bounded set.

Let $Z(\vartheta, t)$ denote the fundamental Cauchy matrix of solutions to (4), and $Z_k(\vartheta, t)$ be a submatrix consisting of k lines corresponding to the chosen coordintes of the vector x. Making the substitution

$$y(t) = Z_k(\vartheta, t)x(t)$$

we obtain the equivalent game of order k [1]:

$$\dot{y} = Z_k(\vartheta, t)\varphi(t, u, v), \ y \in R^k, \ u \in P, \ v \in Q \tag{5}$$

with terminal set M. The sections $\mathbf{W}_t^0$ and W_t^0 of the maximal stable bridges for games (4) and (5) are connected by the relation

$$\mathbf{W}_t^0 = \{x \in R^n : Z_k(\vartheta, t)x \in W_t^0\} \quad .$$

The convexity of the set M implies the convexity of the sections $\mathbf{W}_t^0$ and W_t^0.

Consider a subdivision Γ_m of the segment $[t_*, \vartheta]$. Define

$$D(t_{i+1}, u, v) = Z_k(\vartheta, t_{i+1})\varphi(t_{i+1}, u, v) \quad ,$$

and approximate the convex set M by a polytope M^*. Replace the compact sets P and Q by collections of finite points P^* and Q^*. Let μ^* and ν^* be probability measures on P^* and Q^*, respectively, and $\{l_s\}$ be the net of unit vectors in R^k.

It is known that in the case of a quasi-linear system (5) with a convex terminal set we can take an infinite ball instead of O in relations (2), (3). Applying relations (2), (3) to system (5), we have that the section W_{t_i} which approximates the section $W_{t_i}^0$ is the intersection of the half-spaces

$$\wedge_i^s = \{y \in R^n : l_s' y \le \eta_i(l_s)\}, \ l_s \in \{l_s\} \quad .$$

Here

$$\eta_i(l_s) = \rho(l_s, W_{t_{i+1}}) + (t_{i+1} - t_i)\xi(l_s, t_i)$$

$$\xi(l_s, t_i) = \max_{\mu^* \in \{\mu^*\}} \min_{\nu^* \in \{\nu^*\}} \int_{P^*} \int_{Q^*} (-l_s')D(t_i, u, v)\mu^*(du)\nu^*(dv) \quad ,$$

and $\rho(l_s, W_{t_{i+1}})$ denotes the value of the support function of the set $W_{t_{i+1}}$ on the vector l_s.

The procedure outlined above has been formulated as a computer program for the $k = 2$. The graphs of the sections $W_{t_i}^0$ calculated for a concrete differential game are given below.

Let us consider the following system:

$$\frac{d^2 z}{dt^2} = L(v)u \quad ,$$

$$z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \; L(v) = \begin{pmatrix} \cos v & -\sin v \\ \sin v & \cos v \end{pmatrix}, \; u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \quad . \tag{6}$$

Equation (6) describes the motion of a point of unit mass on the plane (z_1, z_2) under the action of a force $h(t)$. The force $h(t)$ has the same absolute value as the control vector $u(t)$ but forms an angle $v(t)$ with this vector. We assume that the control vector $u(t)$ at each time t may be selected from a given set P consisting of four vectors:

$$u^{(1)} = (1,0)\,,\; u^{(2)} = (0,1)\,,\; u^{(3)} = (-1,0)\,,\; u^{(4)} = (0,-1) \quad .$$

The angle between the force and the control at time t may take any value from the segment $Q = [-\beta, \beta]$, where $0 < \beta < \pi/2$.

Let the performance index be given by

$$\gamma = \|z(\vartheta)\| = (z_1^2(\vartheta) + z_2^2(\vartheta))^{1/2} \quad . \tag{7}$$

The aim of the first player (who governs the control u) is to move the point as close as possible to the origin at time ϑ.

Set $x_1 = z_1$, $x_2 = z_2$, $x_3 = \dot{z}_1$, $x_4 = \dot{z}_2$. Making the subdivisions $y_1 = x_1 + (\vartheta - t)x_3$, $y_2 = x_2 + (\vartheta - t)x_4$ leads to the system

$$\dot{y} = (\vartheta - t)L(v)u \tag{8}$$

with performance index

$$\gamma = \|y(\vartheta)\| = (y_1^2(\vartheta) + y_2^2(\vartheta))^{1/2} \quad . \tag{9}$$

In our example the order of the equivalent game (5) is equal to 2 and the dynamics of the game are described by equation (8). Take a level set of function (9) as the terminal set for this game. Note that it is a circle.

This differential game has been simulated on a computer, taking $\beta = \pi/6$, $\vartheta = 4.8$. Figure 1 shows sections W_t^0 of the set W^0 at times $t = 0, 1.8, 3.3$. Here the radius of the terminal set is 2.45.

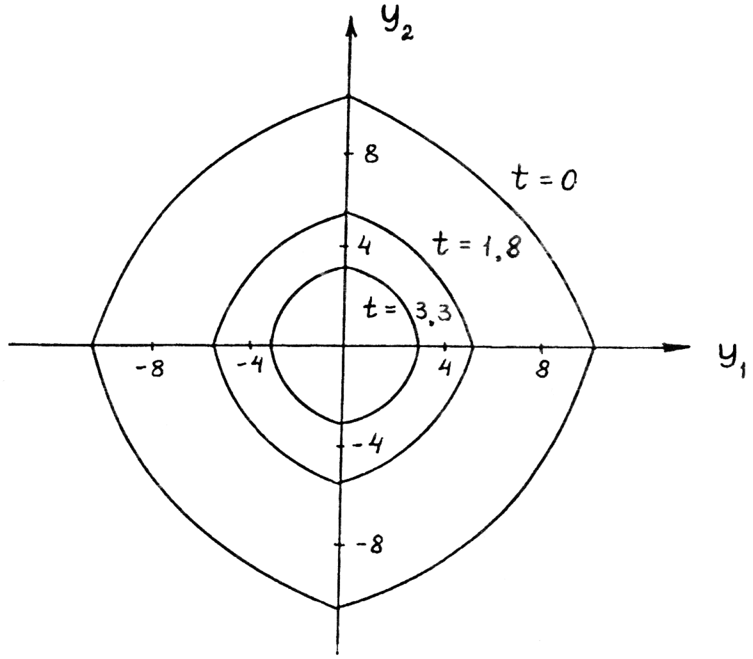

Figure 1. Sections of the set W^0 at times $t = 0$, $t = 1.8$, $t = 3.3$.

The optimal mixed strategies and corresponding motions were also calculated. We note again that in our game the first player tries to minimize the performance index (9), while the second tries to stop him. It is shown that our game satisfies the generalized regularity condition [3], which allows us to use the method of program construction [1].

Mixed strategies which lead to solutions have been constructed according to a scheme using a unified guide. The motion of the unified guide is such that it lies on the appropriate stable bridge. The corresponding control may be found from an extremum condition leading to the position of the guide.

The trajectory generated by optimal mixed strategies on the part of both players from the initial point $z_1(0) = 3.07$, $z_2 = 5.57$, $\dot{z}_1(0) = 0.8$, $\dot{z}_2(0) = 0.2$ on the plane (z_1, z_2) is illustrated in Figure 2. The calculated value of the performance index (7) is

2.47, i.e., approximately equal to the value of the game at the initial point.

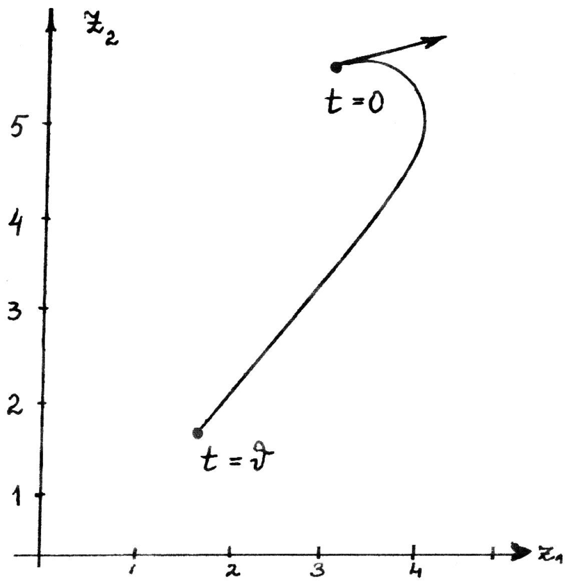

Figure 2. A trajectory generated by optimal mixed strategies for both players.

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