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Construction of the Solvability Set in a Problem of Guiding an Aircraft under Wind Disturbance

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Abstract—A nonlinear system of the fourth order is used for simplified description of the aircraft motion in the horizontal plane under wind disturbance. The aircraft control vector has two components constrained in modulus. One component affects the velocity value, and the second one defines variation of the direction of the velocity vector. The maximal value of the wind disturbance is given. The problem of guaranteed guidance of an aircraft from the initial position to a given terminal set at a fixed instant is considered. The motion is subject to the phase constraints at the intermediate instants. Based on the game theory, an algorithm for backward construction of the solvability set in a problem of guaranteed guidance is proposed. Numerical simulation results of the solvability set construction are described.

INTRODUCTION

In aircraft collision avoidance systems, the problem of aircraft guidance from an initial position to a given area at a prescribed instant is topical. Moreover, at some intermediate instants, the aircraft trajectory must pass through other given spatial areas. The sequence of such instances and areas along the trajectory of a standard maneuver are given by an operator of the air traffic control system for providing safe fly-by of conflicting airplanes.

It would be useful to have some fast computational algorithm analyzing possibility of aircraft guidance through the given areas but under presence of wind disturbance.

In the paper such an algorithm is suggested for the case of aircraft motion in the horizontal plane. An interval of possible values of wind disturbance speed is known. The aircraft motion is described by an ordinary differential equation system of the fourth order. Two phase coordinates have the meaning of the geometric position of the aircraft, the third coordinate is the direction of its velocity vector, and the fourth is the aircraft velocity value. Thus, constructions are implemented in the four-dimensional phase space.

The problem under consideration is interpreted as a problem of finding the solvability set in a differential game of guidance [1, 2] to the terminal set at the fixed instant under phase constraints given at some intermediate instants. To solve the problem, the backward procedure is used based on algorithms for constructing the attainability sets, operations of intersection and forming the convex envelopes of sets. A similar procedure (but in the direct time) was used earlier [3, 4] for

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constructing the informational sets in a problem of aircraft tracking in the horizontal plane under disturbed measurements of its geometric position.

1. PROBLEM FORMULATION

Assume that the aircraft motion in the horizontal plane (Fig. 1) is described by the following ordinary differential equation system:

$$\begin{aligned} \dot{x} &= V\cos\varphi + v_1, \\ \dot{y} &= V\sin\varphi + v_2, \\ \dot{\varphi} &= \frac{ku}{V}, \\ \dot{V} &= w, \qquad\qquad V \geq c > 0. \end{aligned} \tag{1.1}$$

Here, x, y are coordinates of the aircraft geometric position; v_1, v_2 are components of the wind disturbance velocity vector v; φ is the angle between the aircraft velocity vector and the X-axis (heading); V is the aircraft velocity value (in the undisturbed medium); k is the value of the maximal lateral acceleration of the aircraft; and u, w are controls. It is assumed that the controls u, w and the disturbance v are subject to the geometric constraints

$$|u| \leq 1, \quad |w| \leq \mu, \quad v \in Q \subset \mathbb{R}^2,$$

where Q is a compact convex set.

In spite of the fact that system (1.1) is simplified (there are no factors connected with influence of the wind disturbance on the aerodynamics of motion), it is widely applied (see, for example, [5]) in standard navigational computations for constructing reference trajectories, flight plans, etc. Mathematical works are known [6, 7] where system (1.1) was used in the case $v = 0$ for solving the optimal control problems related to motion of an aircraft or a car.

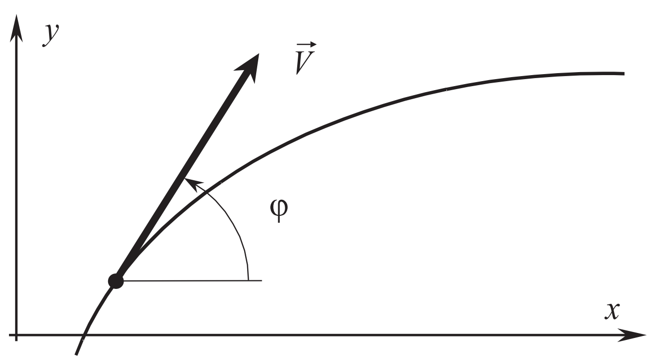

Fig. 1. The coordinate system.

We identify the values of the phase coordinate φ whose difference is an integral multiple of 2π. Define $\mathbb{R}_c = \{V\colon V \geq c\}$.

Let

$$\Phi = \mathbb{R}^2 \times [-\pi, \pi) \times \mathbb{R}_c$$

be the phase space of the problem.

We shall say that the controls u, w belong to the first player, and the vector disturbance v belongs to the second player.

The objective of the first player is to bring the phase vector of system (1.1) to the given terminal set $M \subset \Phi$ at the fixed instant T satisfying the phase constraints $H(\tilde{t}_j) \subset \Phi$ at the given intermediate instants $\tilde{t}_j, j = \overline{1, \omega}$, for any behavior of the second player. We assume that

$$\begin{aligned} &M = M^{\#} \times M^{\diamondsuit}, \\ &t_0 \le \tilde{t}_1 < ... < \tilde{t}_j < ... < \tilde{t}_\omega \le T, \\ &H(\tilde{t}_j) = H^{\#}(\tilde{t}_j) \times H^{\diamondsuit}(\tilde{t}_j), \quad j = \overline{1, \omega}. \end{aligned}$$

Here, $M^{\#}$ and $H^{\#}(\tilde{t}_j)$ are the convex compact sets in the plane x, y; $M^{\diamondsuit}$, $H^{\diamondsuit}(\tilde{t}_j) \subset [-\pi, \pi) \times \mathbb{R}_c$ are the convex compact sets in the plane φ, V.

In the space $[t_0, T] \times \Phi$ it is necessary to construct the maximal set W, from which such guidance is possible. Using terminology of [1], we can say that the set W is the *maximal stable bridge* in the differential game of guidance to the terminal set M.

Clarify now possible application of the computational program for constructing the set W. Assume that an operator of the air traffic control system decided to change the route of an aircraft. To do this, some standard maneuver in the plane x, y is appointed.

For example in Fig. 2, the S-shaped maneuver is shown (the solid curve). The operator estimates how the chosen maneuver can be realized beginning from the initial position (x_0, y_0) at the instant t_0 with the initial values φ_0 and V_0.

The calculation is implemented taking into account the constraints $|u(t) \le 1$, $|w(t)| \le \mu$, and under the assumption of no wind disturbance. Further, on the chosen trajectory the operator defines the terminal set M, connected with the terminal instant T, and the phase constraints $H(\tilde{t}_j)$, $j = \overline{1, \omega}$. By this, the tolerances on the aircraft deviations from the standard trajectory are defined.

In Fig. 2 projections $H^{\#}(\tilde{t}_j)$ of the sets $H(\tilde{t}_j)$ and projection $M^{\#}$ of the terminal set M on the plane x, y are shown in dashes.

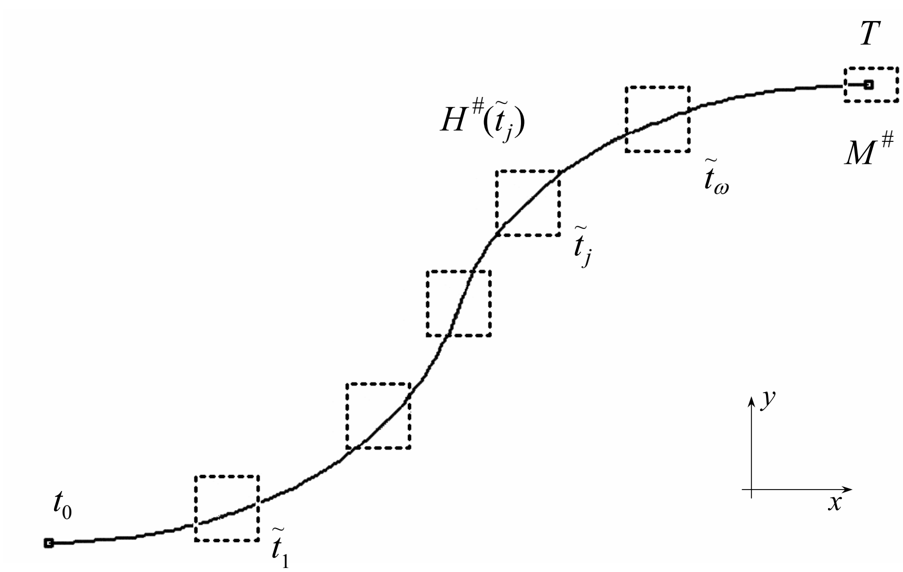

Fig. 2. Projections of the standard trajectory, phase constraints, and the terminal set in the plane x, y.

The operator is interested in possibility of the aircraft flight near the given trajectory, but in the presence of wind disturbance. Imposing the constraint Q on the wind disturbance (for example, in the form $|v| \leq \nu$), the operator calculates the set W with the help of a computational program. If the section $W(t_0)$ of the set W that corresponds to the initial instant t_0 contains the point $(x_0, y_0, \varphi_0, V_0)$, then the choice of aircraft maneuver is correct. Otherwise, the maneuver is not satisfactory and needs changing.

2. BACKWARD PROCEDURES FOR CONSTRUCTING THE SOLVABILITY SET: GENERAL CONSIDERATIONS

In the differential game theory, significant experience was accumulated [8–19] for numerical construction of the solvability set in the game problem of guidance to the given set M at the fixed instant T. As shown in Section 1, the solvability set is also called the *maximal stable bridge*. The algorithms for the backward construction of the maximal stable bridges both for linear problems and for problems with nonlinear dynamics were elaborated (see, for instance, [12–15, 18, 19]).

Dimension of the phase vector in system (1.1) is four. For such a dimension, realization of general algorithms is hampered by significant difficulties. However, the features of system (1.1) taken into account allow us to overcome these difficulties.

Describe the scheme of the backward procedure applied. The numerical construction of cross sections $W(t_i)$ of the maximal stable bridge W is carried out on the given grid of the instants t_i, $i = \overline{0, N}$, $t_0 < t_1 < ... < t_i < ... < t_N = T$. It is convenient to assume that the collection of instants $\{\tilde{t}_j\}$, at which the phase constraints are given, is included into the collection $\{t_i\}$.

The passage from the cross section $W(t_{i+1})$ to the cross section $W(t_i)$ is realized by the following operations:

(A) construction of the attainability set $G(t_i\,;\, t_{i+1}, W(t_{i+1}), v)$ of the considered system in the back time at the instant t_i with the instant t_{i+1} as the initial, the initial set $W(t_{i+1})$, and under fixed constant control v of the second player;

(B) intersection of the obtained sets over $v \in Q$, and intersection with the phase constraint $H(t_i)$

$$W(t_i) = \Big(\bigcap_{v \in Q} G(t_i\,;\, t_{i+1}, W(t_{i+1}), v)\Big) \bigcap H(t_i). \tag{2.1}$$

Here, we assume that $H(t_i) = \mathbb{R}^4$ if at the instant t_i the phase constraint is absent, i.e., $t_i \notin \{\tilde{t}_j\}$.

If the system under investigation were linear on the phase variable, then using convexity of the terminal set M and convexity of the phase constraints $H(\tilde{t}_j)$, $j = \overline{1, \omega}$, we would find the attainability sets $G(t_i\,;\, t_{i+1}, W(t_{i+1}), v)$, $v \in Q$ to be convex as well. Thus, we could work with operations of intersection of convex sets.

Intersection of convex sets is a significantly simpler operation in comparison with operation of intersection of nonconvex sets. Moreover, in the absence of phase constraints in problems with linear dynamics, we intersect the sets $G(t_i\,;\, t_{i+1}, W(t_{i+1}), v)$, $v \in Q$ that differ from each other only by a shift, which is stipulated by various $v \in Q$. It also allows us to simplify the intersection procedure.

The nonlinear system (1.1) has one special property that allows us to use constructively the mentioned simplifications, which are distinctive for linear systems.

3. BACKWARD PROCEDURE FOR CONSTRUCTING THE SOLVABILITY SET IN THE GUIDANCE PROBLEM FOR SYSTEM (1.1)

The mentioned feature of system (1.1) consists in the fact that the variables x and y describing the geometric position of the aircraft do not enter the right-hand side of the system. When integrating the system motion, this allows one to consider the last two equations independently of the first two. Moreover, for fixed functions $\varphi(\cdot)$ and $V(\cdot)$, the first two equations represent a rather simple variant of linear dynamics. The first player's control is absent here.

If in the time interval from the instant t_{i+1} to the instant t_i the second player's control is given as a function of time $v(\cdot)$ (in particular, as a constant), then the set of positions in the plane x, y that was given at the instant t_{i+1} will be directly shifted by some vector, which is a result of integration of the right-hand side under given functions $\varphi(\cdot)$, $V(\cdot)$ and control $v(\cdot)$.

The cross sections $W(t_i)$ of the maximal stable bridge W will be constructed on the time grid t_i with sufficiently small step $\Delta = t_{i+1} - t_i$, $t_i < t_{i+1}$. The set Q is substituted by a convex polygon with a finite number of vertices. The sets $M^{\#}$ and $M^{\diamondsuit}$, $H^{\#}(\tilde{t}_j)$ and $H^{\diamondsuit}(\tilde{t}_j)$ are also substituted by convex polygons.

Due to the discretization, the cross sections $W(t_i)$ are constructed approximately. Moreover, in the process of constructing the next cross section, we use *convexification* of some sets. This leads to roughening from above the cross section under construction. (Detailed computer analysis of influence of this additional convexification procedure shows that such roughening is "not too large.") To emphasize the approximate character of construction and application of operations that roughen the result from above, we use the symbol $\mathbf{W}(t_i)$ instead of $W(t_i)$ to designate the cross sections that are constructed by the suggested algorithm.

3.1. Representation of four-dimensional sets. Description of the algorithm for constructing the sets $\mathbf{W}(t_i)$ is preceded by considering the technique of representation of the four-dimensional sets.

A rectangular grid of nodes $\{\varphi_n\} \times \{V_m\}$ that does not depend on time is introduced in the plane φ, V. On the coordinate φ the grid is built in the interval $[-\pi, \pi]$. On the coordinate V the step of distribution of the grid nodes is given on the basis of the set $M^{\diamondsuit}$ taking into account the rough estimate of possible variation of the velocity value V.

Let

$$L(T) = \{(n,\, m)\colon\ (\varphi_n,\, V_m) \in M^{\diamondsuit} \bigcap H^{\diamondsuit}(T)\},$$

$$F_{n,m}(T) = M^{\#} \bigcap H^{\#}(T), \quad (n,\, m) \in L(T).$$

It is seen that the set $F_{n,m}(T)$ is the same for each $(n,\, m) \in L(T)$.

The set of nodes $(\varphi_n,\, V_m)$, $(n,\, m) \in L(T)$ together with the related sets $F_{n,m}(T)$ is considered as the set $\mathbf{W}(T)$. Thus,

$$\mathbf{W}(T) = \bigcup_{(n,\, m) \in L(T)} \big(\{(\varphi_n,\, V_m)\} \times F_{n,m}(T)\big).$$

The sets $\mathbf{W}(t_i)$ are given in the same form: the totality of nodes $(\varphi_n,\, V_m)$, $(n,\, m) \in L(t_i)$ in the plane φ, V together with the related convex sets $F_{n,m}(t_i)$ in the plane x, y. The recurrent procedure for constructing the sets $\mathbf{W}(t_i)$ is described in the next subsection.

3.2. Passage from the set $\mathbf{W}(\mathbf{t_{i+1}})$ to the set $\mathbf{W}(\mathbf{t_i})$. Let the set $\mathbf{W}(t_{i+1})$ be constructed. It is represented in the form

$$\mathbf{W}(t_{i+1}) = \bigcup_{(n,\, m)\in L(t_{i+1})} \big(\{(\varphi_n,\, V_m)\} \times F_{n,m}(t_{i+1})\big),$$

where $F_{n,m}(t_{i+1})$ are convex sets.

(A) *Approximate construction of the sets* $G(t_i\,;\, t_{i+1},\, \mathbf{W}(t_{i+1}), v)$.

1. Consider an arbitrary node $(\varphi_n,\, V_m)$ such that $(n,\, m) \in L(t_{i+1})$. Let

$$B(\varphi_n,\, V_m) = \left\{ \Pr\,(\varphi_n - \frac{ku}{V_m}\Delta,\, V_m - w\Delta)\colon\ u \in \{-1,\, 0,\, 1\},\, w \in \{-\mu,\, 0,\, \mu\} \right\}.$$

Here, $\Pr\,(\varphi,\, V)$ is the node of the grid $\{\varphi_n\} \times \{V_m\}$ nearest to $(\varphi,\, V)$. The set $B(\varphi_n,\, V_m)$ approximates with the help of the nodes of the grid $\{\varphi_n\} \times \{V_m\}$ the attainability set of the system

$$\begin{aligned} \dot{\varphi} &= \frac{ku}{V}, \\ \dot{V} &= w, \qquad\qquad |u| \le 1, \quad |w| \le \mu \end{aligned} \tag{3.1}$$

from the point $(\varphi_n,\, V_m)$ at the instant $t_i = t_{i+1} - \Delta$. Under this, step Δ agrees with the grid parameters.

2. By the symbol $D_{n,m}(t_i)$ the numbers of those nodes from the instant t_{i+1} are designated, which "passed" into the node (φ_n, V_m), i.e.,

$$D_{n,m}(t_i) = \{(n^*,\, m^*) \in L(t_{i+1})\colon\ (\varphi_n,\, V_m) \in B(\varphi_{n^*},\, V_{m^*})\}.$$

Let

$$K(t_i) = \{(n,\, m)\colon\ D_{n,m}(t_i) \neq \varnothing.$$

The totality of nodes (φ_n, V_m), $(n,\, m) \in K(t_i)$ is (in the grid approximation) the attainability set of system (3.1) at the instant t_i under the initial set at the instant t_{i+1}, which was composed of the nodes (φ_n, V_m), $(n,\, m) \in L(t_{i+1})$.

3. To each node $(\varphi_n,\, V_m)$, $(n,\, m) \in K(t_i)$, the following union is put into correspondence:

$$A_{n,m}(t_i,\, 0) = \bigcup_{(n^*,\, m^*)\in D_{n,m}(t_i)} \big(F_{n^*,m^*}(t_{i+1}) - \Delta V_{m^*}(\cos\varphi_{n^*}, \sin\varphi_{n^*})'\big).$$

Here, the symbol "$'$" means transposition. The set $A_{n,m}(t_i,\, 0)$ approximates the cross section of the four-dimensional attainability set $G(t_i; t_{i+1}, \mathbf{W}(t_{i+1}), 0)$ of system (1.1) under $\varphi = \varphi_n$ and $V = V_m$ (for the case when $v = 0$). Since each set $F_{n^*,m^*}(t_{i+1})$ in the plane x, y is convex, the set $A_{n,m}(t_i,\, 0)$ is the union of convex sets in this plane.

The convex envelope is constructed:

$$S_{n,m}(t_i, 0) = \mathrm{conv} A_{n,m}(t_i,\, 0). \tag{3.2}$$

Let

$$\mathbf{G}(t_i\,;\, t_{i+1}, \mathbf{W}(t_{i+1}), 0) = \bigcup_{(n,\, m)\in K(t_i)} \big(\{(\varphi_n,\, V_m)\} \times S_{n,m}(t_{i+1}, 0)\big).$$

The set $\mathbf{G}(t_i\,;\,t_{i+1},\mathbf{W}(t_{i+1}),0)$ approximates from above (taking into account the errors of the discretization) the attainability set $G(t_i\,;\,t_{i+1},\mathbf{W}(t_{i+1}),0)$.

4. The mentioned specific properties of system (1.1) allow us to find easily the analogous set $\mathbf{G}(t_i\,;\,t_{i+1},\mathbf{W}(t_{i+1}),v)$ for each $v\in Q$ constant in time. Namely,

$$\mathbf{G}(t_i\,;\,t_{i+1},\mathbf{W}(t_{i+1}),v)=\bigcup_{(n,\,m)\in K(t_i)}\big(\{(\varphi_n,\,V_m)\}\times S_{n,m}(t_{i+1},v)\big),$$

where

$$S_{n,m}(t_i,v)=S_{n,m}(t_{i+1},0)-\Delta v.$$

(B) *Construction of the set* $\mathbf{W}(t_i)$.

5. To find the intersection $\bigcap_{v\in Q}\mathbf{G}(t_i\,;\,t_{i+1},\mathbf{W}(t_{i+1}),v)$, it is sufficient to construct the following intersection for each $(n,\,m)\in K(t_i)$:

$$E_{n,m}(t_i)=\bigcap_{v\in Q}S_{n,m}(t_i,v).$$

Implementation of this operation is reduced to finding some positive-homogeneous function

$$\gamma_{n,m}(\ell,t_i)=\min_{v\in Q}\rho\big(\ell,\,S_{n,m}(t_i,v)\big),\quad \ell\in\mathbb{R}^2, \tag{3.3}$$

where $\rho\big(\,\cdot\,,\,S_{n,m}(t_i,v)\big)$ is the support function of the set $S_{n,m}(t_i,v)$, and to the subsequent construction of the convex envelope $\mathrm{conv}\,\gamma_{n,m}(\cdot,t_i)$. By this, we obtain the support function

$$\rho\big(\,\cdot\,,\,E_{n,m}(t_i)\big)=\mathrm{conv}\gamma_{n,m}(\,\cdot\,,t_i)$$

of the set $E_{n,m}(t_i)$. When realizing the minimum in (3.3), we look through only the vertices of the polygon, which approximates the set Q.

6. Let

$$L(t_i)=\{(n,\,m)\in K(t_i)\colon\ (\varphi_n,\,V_m)\in H^{\diamondsuit}(t_i)\},$$
$$F_{n,m}(t_i)=E_{n,m}(t_i)\cap H^{\#}(t_i),\quad (n,\,m)\in L(t_i).$$

Taking into account (2.1) and the way of representation of the four-dimensional sets, we have

$$\mathbf{W}(t_i)=\bigcup_{(n,\,m)\in L(t_i)}\big(\{(\varphi_n,\,V_m)\}\times F_{n,m}(t_i)\big).$$

We emphasize that precisely on account of the operation of convexification in formula (3.2) do we obtain the convexity of the sets $F_{n,m}(t_i)$. This fact together with application of the grid on coordinates φ, V defines the simplicity of realization of the backward procedure. The operation of convexification expands a little the constructed tube of the solvability set in comparison with the true one.

The algorithm composed of steps 1–6 in many aspects is similar to the one elaborated in [3, 4] for constructing informational sets in a problem of aircraft tracking in the horizontal plane. In [3, 4], comparison of the exact attainability sets of system (1.1) with the sets obtained by the operation of constructing the convex envelope of the union of convex sets was conducted. It was shown there that the errors that appeared are insignificant from the practical point of view.

Remark. Construction of the feedback control that keeps the aircraft inside the solvability set under the wind disturbance is a separate problem and is not considered in this paper.

4. SIMULATION RESULTS

The following numerical data were taken for simulation of the described algorithm:

- the aircraft maximal lateral acceleration $k = 4$ m/s^2;
- the maximal longitudinal acceleration (control) $\mu = 2$ m/s^2;
- the constraint on the wind disturbance is given in the form $|v| \leq \nu$ and a value of the parameter ν is taken from the interval 16 m/s–22 m/s;
- the initial instant $t_0 = 0$ s;
- the whole time of motion $T = 120$ s;
- the aircraft initial position $x_0 = 0$ m, $y_0 = 0$ m, $\varphi_0 = 0°$, $V_0 = 200$ m/s;
- the reference terminal point coordinates at the instant T are $x_T = 15216$ m, $y_T = 8543$ m, $\varphi_T = 0°$, and $V_T = 200$ m/s;
- the reference phase trajectory between the initial position and the reference terminal point is constructed by means of program controls (with a "reserve" on unknown disturbances) $u = 0.8$, $w = -1.5$ m/s^2 at the time interval $[0, 60)$ s and $u = -0.8$, $w = 1.5$ m/s^2 on the interval $[60, 120]$ s;
- the terminal set M has a size ± 300 m in the coordinate x, ± 300 m in the coordinate y, $\pm 20°$ in the angle φ, and ± 20 m/s in the velocity V, and the centers of these intervals coincide with the corresponding values of the reference terminal point;
- the instants for which the phase constraints were given are $\tilde{t}_1 = 20$, $\tilde{t}_2 = 40$, $\tilde{t}_3 = 60$, $\tilde{t}_4 = 80$, and $\tilde{t}_5 = 100$ s;
- the phase constraints $H(\tilde{t}_j)$, $j = \overline{1,5}$, have a size of ± 300 m in the coordinate x, ± 300 m in the coordinate y, $\pm 20°$ in the angle φ, and ± 40 m/s in the velocity V, while the centers of these intervals coincide with the corresponding values of parameters of the reference trajectory at the given instants;
- the time step Δ in the backward constructions is 1 s; the size of the grid in the plane φ, V is 720×64; the sets in the plane x, y are represented by convex polygons with the uniform grid of normals (24 normals).

Dynamics of variation in time of the set $\mathbf{W}(t)$ for the case $\nu = 16$ m/s is shown in Fig. 3. The projections $\mathbf{W}^{\#}(t)$ of the four-dimensional sets $\mathbf{W}(t)$ in the plane x, y are marked by grey. It is seen that the size of the cross section projection increases when moving in the back time from the terminal set. After intersection with the phase constraint, the size of the cross section decreases. Further, the process develops similarly. At the instant $t_0 = 0$, the constructed cross section of the bridge $\mathbf{W}(t_0)$ contains the aircraft initial position.

The structure of the cross section $\mathbf{W}(t_0)$ of the maximal stable bridge at the instant $t_0 = 0$ is illustrated by Fig. 4. Here, the solid curve shows the initial part of the reference trajectory in projection into the plane x, y, small square shows the aircraft position at the initial instant, and the dashed square shows the first phase constraint at the instant $\tilde{t}_1$. The projection $\mathbf{W}^{\#}(t_0)$ of the four-dimensional set $\mathbf{W}(t_0)$ into the plane of geometrical coordinates x, y is shown in light grey. Its

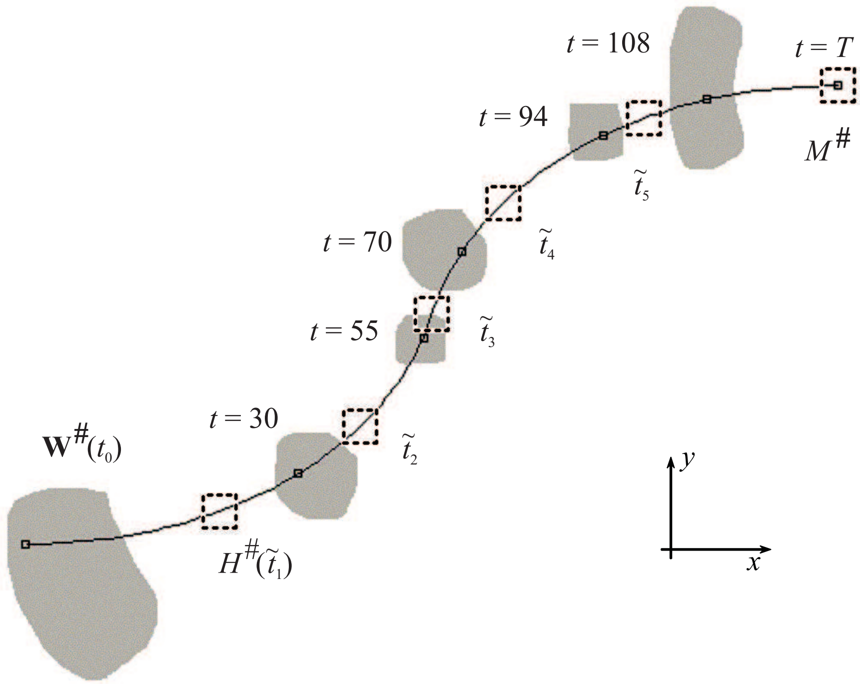

Fig. 3. Dynamics of variation in time of the set $\mathbf{W}^{\#}(t)$.

whole contour is seen. Inside this projection the projection of the three-dimensional layer, which corresponds to the grid node $\{V_m\}$ maximally close to the aircraft initial velocity $V_0 = 200$ m/s, is marked in white. Inside the projection of this layer, the two-dimensional layer, which corresponds to the grid node $\{\varphi_n\}$ maximally close to the initial value of the angle $\varphi_0 = 0°$, is shown in dark grey.

Figure 5 shows images of the three-dimensional layers of the four-dimensional set $\mathbf{W}(t_0)$. The layers correspond to five different values of the velocity and are marked in grey of various depth.

Influence of the level ν of the wind disturbance maximal value on the size of the projection $\mathbf{W}^{\#}(t_0)$ is illustrated in Fig. 6. Here, the solid curve marks the initial part of the reference trajectory, the small square shows the aircraft position at the initial instant, and the dashed square shows the phase constraint at the instant $\tilde{t}_1$.

In Fig. 7, the projection $\mathbf{W}^{\#}(t_0)$ of the set $\mathbf{W}(t_0)$ in the plane φ, V is shown for the same data.

As was mentioned above, the aircraft initial position belongs to the set $\mathbf{W}(t_0)$ under the constraint $\nu = 16$ m/s (Figs. 6a, 7a). Therefore, the problem of guiding the aircraft to the terminal set can be solved under this constraint on the wind disturbance values. But under the constraint $\nu = 20$ m/s (Figs. 6b, 7b) the cross section of the bridge does not reach the aircraft initial position. This means that guidance of the aircraft from the initial position to the given terminal set is not guaranteed. Under the constraint $\nu = 21$ m/s (Figs. 6c, 7c) the cross section $\mathbf{W}(t_0)$ of the bridge at the initial instant is placed at a larger distance from the initial position of the aircraft and, moreover, significantly decreases in size. Under further growth of the constraint ν, the set $\mathbf{W}(t_0)$ becomes empty.

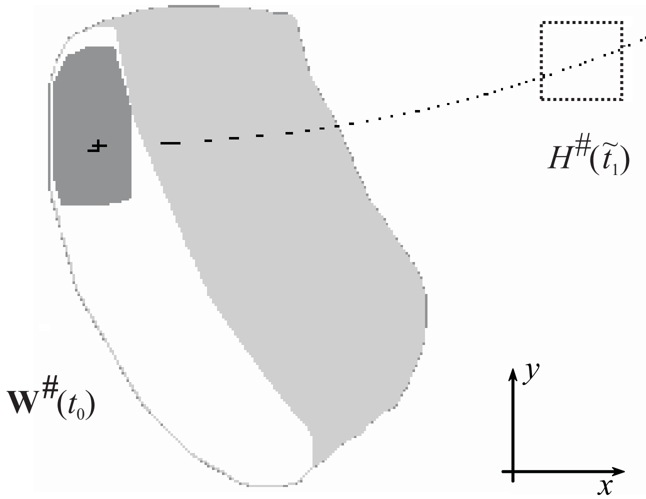

Fig. 4. The structure of the four-dimensional set $\mathbf{W}(t_0)$ by layers in V and φ (as projections in the plane x, y), the constraint on the wind value is $\nu = 16$ m/s.

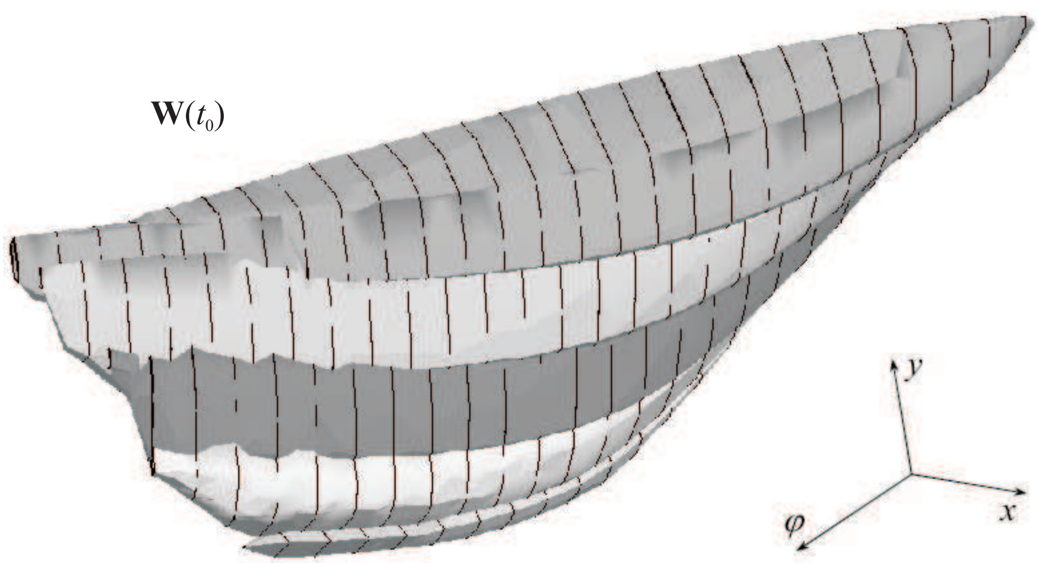

Fig. 5. The three-dimensional image of the four-dimensional set $\mathbf{W}(t_0)$, the constraint on the wind value is $\nu = 16$ m/s.

The size and structure of the set $\mathbf{W}(t_0)$ also depends significantly on the instants at which the phase constraints are given and on the sizes of these constrains.

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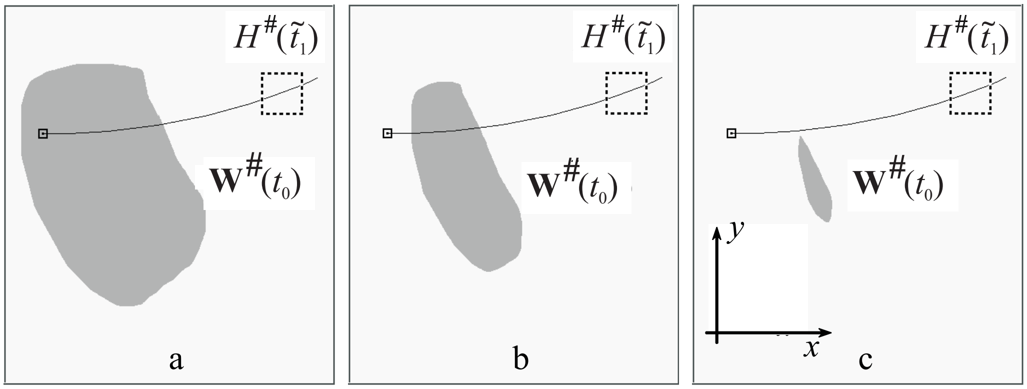

Fig. 6. Comparison of the sets $\mathbf{W}^{\#}(t_0)$ for various values of the constrains on the wind: a) $\nu = 16$ m/s, b) $\nu = 20$ m/s, c) $\nu = 21$ m/s.

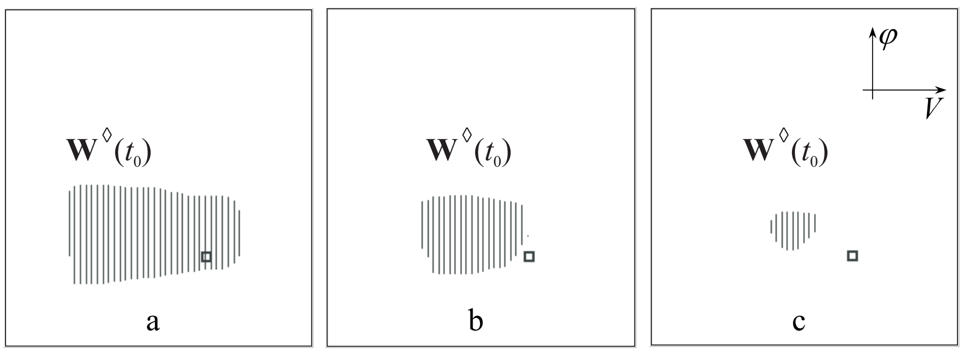

Fig. 7. Comparison of the sets $\mathbf{W}^{\diamond}(t_0)$ for various values of the constrains on the wind: a) $\nu = 16$ m/s, b) $\nu = 20$ m/s, c) $\nu = 21$ m/s.

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