Invariant Equilibria of Linear Discrete-time Games

Paolo Caravani
Department of Electrical Engineering
University of L’Aquila
Monteluco di Roio, 67040 L’Aquila, Italy.
caravani@ing.univaq.it

Abstract

The paper studies a linear dynamic game in discrete-time driven by bounded additive noise and \( r \) independent agents having full information on the state but no information over rival strategies. Players act independently with strategies taking values in closed bounded sets \( U_i \) that are not common knowledge. Payoffs represent set membership rather than utility: the goal of player \( i \) is to keep the state in a bounded set \( \Omega_i \). This possibility depends upon the initial state, the strategies of other players, the noise amplitude and of course the own adopted strategy. Such a framework gives rise to an extension of the notion of robust-controlled-invariance, which is termed invariant equilibrium. We characterize invariant equilibria and show that for their existence it is necessary that the free dynamics be stable. In the case of ellipsoidal sets, existence conditions can be further specified in terms of linear matrix inequalities. Equilibrium strategies are of linear state-feedback type, and their computation is extremely efficient. Applications in finance are discussed.

work partially supported by European Commission under STREP project TREN/07/FP6AE/S07.71574/037180 IFLY.

I. INTRODUCTION

We consider an \( r \)-person game with dynamics described by

\[
\begin{align*}
x^{+} &= Ax + B_1 u_1 + \cdots + B_r u_r + w \\
        &\in W \subset \mathbb{R}^n \\
u_1 &\in U_1 \subset \mathbb{R}^{m_1} \\
     \vdots \\
u_r &\in U_r \subset \mathbb{R}^{m_r}
\end{align*}
\]

where, as a standing assumption, we take \( W, U_i \) to be bounded sets with \( 0 \in W \) and \( U_i \) symmetric (\( u \in U_i \Rightarrow -u \in U_i \)). The state \( x \in \mathbb{R}^n \) is known to all players. Each player selects a control action independently of other players knowing \( x \) and the system matrices. Each player \( i \) strives to keep \( x \) inside \( \Omega_i \), a bounded subset of \( \mathbb{R}^n \).

Definition 1: An Invariant Equilibrium (IE) is a collection \( E = \{(\Omega_i, U_i), \ i \in [1,r]\} \) of sets satisfying the property

\[
\forall x \in \Omega_i, \ \exists u_i \in U_i : \ x^{+} \in \Omega_i, \ \forall u_j \in U_j, \ \forall w \in W.
\]

Thus in an IE, if the initial state \( x_0 \) belongs to \( \Omega_i \) player \( i \) is able to satisfy his own state and control constraints regardless of what other players do within their own control constraints. Readers accustomed to standard formalism may object to the lack of a payoff in the above definition. Indeed, payoff functions are replaced here by sets. Standard formalism could be restored by introducing indicator functions \( I(\Omega_i) \) qualifying a player’s effort to keep the state inside \( \Omega_i \) as a success (\( I = 1 \)) or a failure (\( I = 0 \)). But, as argued in [3], the advantage would be purely formal, with all interesting strategic aspects remaining unchanged.
A. strategic outlook

Notice that definition 1 does not imply state constraints should be satisfied for all players. In other words, it is not required that the $\Omega_i$’s have a non-empty common, or even pairwise, intersection. Examples of IE with disjoint $\Omega_i$’s have been exhibited in [4]. In that case ”winning” players are those whose constraint set $\Omega_i$ includes the initial state. On the other hand, if a common intersection exists the strategic character of the game changes: all players can ”win”. The distinction is closely reminescent of the difference, in two person games, between the zero-sum and the non-zero sum case, the former corresponding to disjoint sets, the latter corresponding to non-empty intersection. Here we deal with the case of common non-empty intersection, thus allowing for possible cooperation. The main strategic aspect concerns information: players are not informed of the actions taken by other players. In addition, they may or may not know the control constraints $U_i$ other players are facing. As a result, actions acquire a double role: active, when used as an instrument by a player; passive, when perceived as a disturbance by other players – a disturbance whose amplitude, if unknown, could be deduced by the awareness of each player about this double role.

If the control constraints $U_i$ are known and common knowledge, then IE exhibits the self-enforcing property of Nash equilibrium. Namely, as long as each player $j \neq i$ satisfies $u_j \in U_j$ with $(\Omega_j, U_j) \in E$, it is beneficial to player $i$ to use $u_i \in U_i$ as this enables him to keep $x$ in $\Omega_i$ (assuming $x_0 \in \Omega_i$). But if player $i$ deviates by playing $u_i \notin U_i$, then some of the other players may be unable to keep the state within $\Omega_j$ and this may result in $x \notin \Omega_i$, e.g. player $i$ is no longer guaranteed to be able to make $\Omega_i$ robust invariant.

The situation is somewhat more difficult if constraints $U_i$ are not known to all players. Due to non-uniqueness of IE (as shown momentarily) coordination appears necessary to select one shared IE, unless a collective rationality principle can be invoked. Among the many available, we find the Nash bargaining solution to be quite fit to the present case. The rationality embodied in the Nash bargaining solution can bridge the gap between the decentralized information available to the players and the centralized information that would be required to evaluate a particular IE strategy.

B. a necessary condition

In this framework it is possible to give a general condition for existence of an IE. For completeness, we include a proof of this previously published result [4]

**Theorem 2**: An IE exists only if $A$ is a stable matrix.

*Proof:* Since we are dealing with necessity, it’s enough to take $r = 2$. Let $x_i \in \Omega_i, i = 1, 2$. By definition of IE,

\[\exists u_1 \in U_1 : \ Ax_1 + B_1 u_1 + B_2 U_2 \oplus W \subset \Omega_1\]
\[\exists u_2 \in U_2 : \ Ax_2 + B_2 u_2 + B_1 U_1 \oplus W \subset \Omega_2.\]

As $0 \in W$, inclusions must hold with $W$ removed

\[\exists u_1 \in U_1 : \ Ax_1 + B_1 u_1 + B_2 U_2 \subset \Omega_1\]
\[\exists u_2 \in U_2 : \ Ax_2 + B_2 u_2 + B_1 U_1 \subset \Omega_2.\]

Letting $\Omega = \Omega_1 \oplus \Omega_2$ and $U = B_1 U_1 \oplus B_2 U_2$, the above can be written

\[\forall \hat{x} \in \Omega \ \exists \hat{u} \in U : \ A\hat{x} + \hat{u} + U \subset \Omega\]
Since $U$ is symmetric it contains $u = -\hat{u}$, therefore

$$A\Omega \subset \Omega.$$  

that is, $A$ is positively invariant on $\text{Conv hull}(\Omega \cup -\Omega)$, a bounded set containing the origin, which implies $A$ must be stable. 

Sufficient conditions on the other hand are either trivial or not very informative. For example, it is easy to prove

**Lemma 3:** If $A$ is a Schur matrix, then the collection of sets $E_0 = \{(\Omega_i, U_i), i \in [1, r]\}$ with

\[
\begin{align*}
U_i & = \{0\} \quad \forall i \\
\Omega_i & = \Omega_{\infty} \quad \forall i \\
\Omega_{\infty} & = W \oplus AW \oplus A^2W \oplus \ldots
\end{align*}
\]

is an invariant equilibrium. 

**Proof:** Since $A$ is Schur and $W$ is bounded, $\Omega_{\infty}$ is bounded and it is immediate to verify that $A\Omega_{\infty} \oplus W = \Omega_{\infty}$. 

We shall call $E_0$ a trivial IE. What the example shows is, unless sets $\Omega_i, U_i$ are further characterized by additional properties (for example, $U_i \neq \{0\}$) the IE notion can be too generic to be interesting.

In [4] the sets $\Omega_i$ were parametrized in a family of polyhedra and results were given for the two-player case. In [5] the $\Omega_i$ were assumed to be confined inside prescribed convex sets but were left otherwise unspecified and an algorithm to determine a maximal IE in two-person games was established. In [6] a more general polyhedral parametrization permitted to extend results to the case of $r$-person games via linear programming. In all of the above, equilibrium strategies resulted in non-linear feedback, thus requiring rather intensive on-line computation. Here we assume each player has an input-output constraint of the form

$$u_i \in U_i \subset \mathbb{R}^{m_i}; \quad y_i \in Y_i \subset \mathbb{R}^{p_i}, \quad y_i = C_i x, \quad i = 1 \ldots r$$

that should be satisfied for all $x \in \Omega_i$ and assume $\Omega_i$ to be parametrized in the family of “sums of ellipsoids”. In this case state-feedback strategies will turn out to be linear, e.g. $u_i = K_i x$ for $x \in \Omega_i$.

II. **ANALYTICAL TOOLS**

A. **robust invariance**

Consider

$$x^+ = Ax + w, \quad w \in W. \quad (3)$$

We recall

**Definition 4:** A set $\Omega$ is robust invariant with respect to $W$ (W-RI, for short) for (3) if

$$\forall x \in \Omega, \quad Ax + w \in \Omega, \quad \forall w \in W. \quad (4)$$

Let $\text{Pre} \Omega$ denote the set of $x \in \mathbb{R}^{n}$ reaching $\Omega$ in one step, i.e. $x \in \text{Pre} \Omega \Leftrightarrow Ax \in \Omega$. Robust invariant sets are fully characterised as Minkowski sums
Lemma 5: $\Omega$ is $W$-RI for (3) if and only if $\Omega = \Sigma \oplus W$ and
\[
\Sigma \oplus W \subset \text{Pre} \Sigma. \tag{5}
\]

Proof: (if) By definition of $\text{Pre}$ inclusion (5) implies $Ax \in \Sigma$ for all $x \in \Sigma \oplus W$. If $Ax \in \Sigma$ then $Ax + w \in \Sigma \oplus W$ and we conclude
\[
\forall x \in \Sigma \oplus W, \ Ax + w \in \Sigma \oplus W, \ \forall w \in W
\]
which is (4) with $\Omega = \Sigma \oplus W$.

(only if) Assume $\Omega$ is $W$-RI and let
\[
\Sigma = \{\sigma : \ \sigma = Ax, \ x \in \Omega\}
\]
Then the $W$-RI condition (4) implies
\[
\Sigma + w \in \Omega, \ \forall w \in W \\
\Rightarrow \ \Sigma \oplus W = \Omega
\]
and condition (4) can be re-written
\[
\forall x \in \Sigma \oplus W, \ Ax + w \in \Sigma \oplus W, \ \forall w \in W \\
\forall x \in \Sigma \oplus W, \ Ax + W \subset \Sigma \oplus W \\
\forall x \in \Sigma \oplus W, \ Ax \in \Sigma \\
\forall x \in \Sigma \oplus W, \ x \in \text{Pre} \Sigma
\]
that is (5).

Remark 6: We notice that if $0 \in \Omega$, then in order for $\Omega$ to be $W$-RI it is necessary that $W \subset \Omega$.

For the rest of the paper we assume $0 \in \Omega$.

Lemma 7: Let $0 \in \text{int} W$. In order for a bounded $\Omega$ to be $W$-RI for (3) matrix $A$ must be Schur.

Proof: Since $\Omega$ is bounded, matrix $A$ cannot be unstable so either $A$ is Schur or it is marginally stable, e.g. $A$ has one eigenvalue $\lambda$ of multiplicity 1 with $|\lambda| = 1$. We show that the latter case is impossible. Consider the $W$-RI condition (5) in the form
\[
\forall x \in \Sigma \oplus W, \ Ax \in \Sigma. \tag{6}
\]
Since $W$ has a nonempty interior, there exists an $x_0 \in \Sigma \oplus W$ and a neighborhood of it
\[
\mathcal{N} = \{x : ||x_0 - x|| < \epsilon\}
\]
such that $\mathcal{N} \subset \Sigma \oplus W, \mathcal{N} \cap \Sigma = \emptyset$. Consider the evolution of $x^+ = Ax$ starting from $x_0$. Since $A$ is marginally stable there exists a $k < \infty$ such that $x_k = A^k x_0 \in \mathcal{N}$. Consider the sequence $\{x_h\} = \{A^h x_0\}$. Either $\forall h \in [1 \ k], \ x_h \notin \Sigma$, in which case $x_0$ contradicts (6); or, there is a maximal $\bar{h} < k$ such that $x_{\bar{h}} \in \Sigma$. But then $x_{\bar{h}+1} = A x_{\bar{h}} \notin \Sigma$, again contradicting (6). Hence $|\lambda| = 1$ is impossible and $A$ must be Schur.

It is easily checked that if two sets are $W$-RI, so is their intersection, and since each $W$-RI set must contain $W$, the minimal $W$-RI is well defined.
**Theorem 8:** The minimal $W$-RI set for $x^+ = Ax + w$, $w \in W$ is

$$
\Omega_\infty = \lim_{x \to \infty} \Omega_k = W \oplus AW \oplus A^2W \oplus \cdots \oplus A^kW.
$$

**Proof:** We first check that $\Omega_\infty$ is $W$-RI. Let $x \in \Omega_\infty$. Then

$$
Ax + w \in A\Omega_\infty \oplus W = AW \oplus A^2W \oplus \cdots \oplus W = \Omega_\infty.
$$

Let now $\Omega = \Sigma \oplus W$ be any other $W$-RI set.

\[
\Sigma \oplus W \subset \text{Pre} \Sigma \iff A\Sigma \oplus AW \subset \Sigma
\]

\[
\Rightarrow A\Sigma \oplus AW \oplus W \subset \Sigma \oplus W = \Omega
\]

\[
\Rightarrow A^2\Sigma \oplus A^2W \oplus AW \oplus W \subset A\Sigma \oplus AW \oplus W \subset \Omega
\]

\[
\cdots \cdots
\]

\[
\Rightarrow A^k\Sigma \oplus A^kW \oplus A^{k-1}W \oplus \cdots \oplus W \subset \Omega
\]

\[
\Rightarrow A^k\Sigma \oplus \Omega_k \subset \Omega
\]

where each $\Rightarrow$ is obtained from the previous multiplying by $A$ and adding $W$. For $k \to \infty$ the first term vanishes because $A$ is Schur. The second tends to $\Omega_\infty$ by definition. Hence $\Omega_\infty \subset \Omega$ for any set $\Omega$ which is $W$-RI for $x^+ = Ax + w$. \hfill \blacksquare

It is well known [9] that $\Omega_\infty$ is an attractive set for (3).

**B. ellipsoidal invariance**

We next particularize the above to the case of ellipsoids.

\[
\Sigma = \mathcal{E}(X) = \{ x \in \mathbb{R}^n : x'X^{-1}x \leq 1 \} \tag{7}
\]

\[
W = \mathcal{E}(P) = \{ x \in \mathbb{R}^n : x'P^{-1}x \leq 1 \} \tag{8}
\]

with $X, P$ symmetric positive definite. Ellipsoidal inclusion is well known to obey

\[
\mathcal{E}(Q) \subset \mathcal{E}(R) \iff Q \leq R \tag{9}
\]

Moreover,

\[
\text{Pre} \mathcal{E}(Q) = \{ x : (Ax)'Q^{-1}(Ax) \leq 1 \} = \{ x : x'A'Q^{-1}Ax \leq 1 \} = \mathcal{E}([A'Q^{-1}A]^{-1}) \tag{10}
\]

Using (5) robust invariance of $\mathcal{E}(X) \oplus \mathcal{E}(P)$ holds if and only if

\[
\mathcal{E}(X) \oplus \mathcal{E}(P) \subset \text{Pre} \mathcal{E}(X). \tag{11}
\]

Since ellipsoids are not preserved under Minkowski sum, the lhs is not an ellipsoid and inclusion (11) is hard to express. However, the sum of two ellipsoids is contained in an ellipsoid that can be “optimally” selected.

**Lemma 9:** see [10, Lemma 2.2.1]

\[
\{ x : x'X^{-1}x \leq 1 \} \oplus \{ x : x'P^{-1}x \leq 1 \} \subset \{ x : x'[(1 + \alpha)X + (1 + \alpha^{-1})P]^{-1}x \leq 1 \}, \quad \forall \alpha > 0 \tag{12}
\]
In [10] it is also proved that $\alpha$ can be selected to minimize the sum of squares of the semiaxes of the ellipsoid on the rhs of (12)

$$\alpha^* = \frac{(\text{tr}X)^{\frac{1}{2}}}{(\text{tr}P)^{\frac{1}{2}}}$$

or its minimal volume

$$\alpha^* = \text{sol.n of:} \quad \sum_{i=1}^{n} \frac{1}{\lambda_i + \alpha} = \frac{n}{\alpha(\alpha + 1)}$$

where $\lambda_i$ is the $i$-th eigenvalue of the pencil $X - \lambda P$. Furthermore, the approximation of the ellipsoidal sum in Lemma (9) is tight in the sense $E_1 \oplus E_2 = \cap_{\alpha > 0} E_{\alpha}$ where $E_1, E_2$ are the ellipsoids on the lhs, and $E_{\alpha}$ on the rhs of (12).

Letting

$$X_\alpha = (1 + \alpha)X + (1 + \alpha^{-1})P$$

condition (5) can then be expressed as ellipsoidal inclusion

$$E(X_\alpha) \subset \text{Pre} E(X)$$

and we have

**Theorem 10:** $E(X) \oplus E(P)$ is $E(P)$-robust invariant for (3) with $W = E(P)$ if and only if there exists a scalar $\alpha > 0$ such that

$$[(1 + \alpha)X + (1 + \alpha^{-1})P]^{-1} - A'X^{-1}A \geq 0$$

with equality holding if $\Omega_{\infty} = E(X) \oplus E(P)$

**Proof:** sufficiency follows from inclusions (12) and (5). Necessity from tightness of condition (12). Now if $\Omega_{\infty} = E(X) \oplus E(P)$ then

$$E(X) = A E(P) \oplus A^2 E(P) \oplus \ldots$$

and it is easy to check that $A(E(X) \oplus E(P)) = E(X)$, that is

$$E(X) \oplus E(P) = \text{Pre} E(X)$$

whence $\text{Pre} E(X) \subset E(X_\alpha) \subset \text{Pre} E(X)$ or, $E(X_\alpha) = \text{Pre} E(X)$ which is (14) with $=.$

Turning now to the system $x^+ = (A + BK)x + w$, we have

**Theorem 11:** The set of all state feedback matrices for which $\Omega = E(X) \oplus E(P)$ is $E(P)$-RI for $x^+ = (A + BK)x + w$ with $\Omega_{\infty} \subset \text{int} \Omega$ is described by

$$K = YX_\alpha^{-1}$$

where the parameters $X, Y, \alpha$ satisfy the LMI

$$\begin{bmatrix}
X & AX_\alpha + BY \\
Y' B' + X_\alpha A' & X_\alpha
\end{bmatrix} > 0, \quad \alpha > 0.$$
Proof: By Lemma (10), \( \Omega \) is \( \mathcal{E}(P) \)-RI if and only if
\[
X^{-1}_\alpha - (A + BK)'X^{-1}(A + BK) > 0, \quad \text{for some } \alpha > 0.
\]
By using Schur complement and standard control arguments [1], this is equivalent to (15).

Finally, it is well known that input-output constraints of the form
\[
\|u\| = \|Kx\| < \gamma \quad ||y|| = \|Cx\| < \eta
\]
are satisfied in all points \( x \in \mathcal{E}(X_\alpha) \) if the following LMI’s hold
\[
\begin{bmatrix}
\gamma^2 I & Y \\
Y' & X_\alpha
\end{bmatrix} > 0
\]
(16)
\[
\begin{bmatrix}
\eta^2 I & CX_\alpha \\
X_\alpha C' & X_\alpha
\end{bmatrix} > 0
\]
(17)
(see [1]). Now constraints satisfied over \( \mathcal{E}(X_\alpha) \) are also satisfied over \( \Omega \) because \( \Omega \subset \mathcal{E}(X_\alpha) \). Therefore LMI system (16,17) together with (15) characterizes all feedback matrices that make \( \Omega \) a \( \mathcal{E}(P) \)-RI set over which input-output constraints are satisfied.

Remark 12: Notice that, strictly speaking, (15) is a LMI only for fixed \( \alpha \), see (13). But since \( \alpha \) is a scalar, feasibility can be checked by a simple one-dimensional search.

Let now \( \mathcal{F} \) be the set of \((\eta^2, \gamma^2, X, Y)\) for which (15,16,17) are feasible for fixed \( \alpha \); and let \( \mathcal{F}_\gamma \) be the set of \((\eta^2, X, Y)\) such that \((\eta^2, \gamma^2, X, Y) \in \mathcal{F}\). Let \( \bar{\mathcal{F}}, \bar{\mathcal{F}}_\gamma \) be their closures. Consider the function
\[
J : \mathbb{R} \mapsto \mathbb{R}, \quad J(\gamma^2) = \min_{\mathcal{F}_\gamma} \eta^2.
\]
Being the minimum of a convex function on a closed convex set \( J \) is well defined and

Lemma 13: The function \( J(\gamma_1^2) < J(\gamma_2^2) \Rightarrow \gamma_1^2 > \gamma_2^2 \).

Proof: Let \( J(\gamma_1^2) < J(\gamma_2^2) \). Then
\[
\forall x \in \mathcal{E}(X_{\alpha,1}), \quad ||Cx||^2 \leq J(\gamma_1^2)
\]
(18)
\[
\forall x \in \mathcal{E}(X_{\alpha,2}), \quad ||Cx||^2 \leq J(\gamma_2^2)
\]
(19)
and \( \mathcal{E}(X_{\alpha,1}) \) contains the set \( \mathcal{E}(X_1) \oplus W \), which is RI under \( ||u||^2 < \gamma_1^2 \); \( \mathcal{E}(X_{\alpha,2}) \) contains the set \( \mathcal{E}(X_2) \oplus W \), which is RI under \( ||u||^2 < \gamma_2^2 \). Expressing (18, 19) as ellipsoidal inclusions we get
\[
\mathcal{E}(X_1) \oplus W \subset \mathcal{E}(X_{\alpha,1}) \subset \mathcal{E}([C'J_1^{-1}(\gamma_1^2)C]^{-1})
\]
\[
\mathcal{E}(X_2) \oplus W \subset \mathcal{E}(X_{\alpha,2}) \subset \mathcal{E}([C'J_2^{-1}(\gamma_2^2)C]^{-1}).
\]
Now
\[
J(\gamma_1^2) < J(\gamma_2^2) \Rightarrow \mathcal{E}([C'J_1^{-1}(\gamma_1^2)C]^{-1}) \subset \mathcal{E}([C'J_2^{-1}(\gamma_2^2)C]^{-1})
\]
\[
\Rightarrow \mathcal{E}(X_{\alpha,1}) \subset \mathcal{E}(X_{\alpha,2})
\]
\[
\Rightarrow \mathcal{E}(X_1) \oplus W \subset \mathcal{E}(X_2) \oplus W.
\]
\( \mathcal{E}(X_1) \oplus W \) cannot be a set which is RI under \( ||u||^2 < \gamma_2^2 \) for otherwise \( J(\gamma_1^2) = J(\gamma_2^2) \). Thus under \( ||u||^2 < \gamma_2^2 \) we are able to make invariant \( \mathcal{E}(X_2) \oplus W \) but not a set that is contained in it like \( \mathcal{E}(X_1) \oplus W \). Since the latter is RI under \( ||u||^2 < \gamma_1^2 \) we conclude \( \gamma_1^2 > \gamma_2^2 \).
III. INVARIANT ELLIPSOIDAL EQUILIBRIA

We assume player $i$ is interested in keeping variable $y_i = C_ix$ inside a set $Y_i$ and faces constraints of the form

$$U_i = \{u_i : \|u_i\| < \gamma_i\}$$
$$Y_i = \{y_i : \|y_i\| < \eta_i\}.$$

We further assume $W = \{w : \|w\| < \rho\}$.

Isolating the action of the $i$-th player, rewrite (1) as

$$x^+ = Ax + B_iu_i + B_{-i}v_{-i}, \quad y_i = C_ix$$

with

$$B_{-i} = \begin{bmatrix} r^{1/2} [B_1\gamma_1 | \ldots | B_{i-1}\gamma_{i-1} | B_{i+1}\gamma_{i+1} | \ldots | B_r\gamma_r | \rho I] \\ v_{-i} = r^{-1/2} \{u_1'/\gamma_1 | \ldots | u_{i-1}'/\gamma_{i-1} | u_{i+1}'/\gamma_{i+1} | \ldots | u_r'/\gamma_r | w'/\rho\}' \end{bmatrix}.$$

Notice that $B_{-i}$ contains $\rho I$, a non-singular matrix of order $n$ for $\rho > 0$, so it has rank $n$. Moreover

$$\|v_{-i}\|^2 = r^{-1} \sum_{k \neq i} ||u_k||^2/\gamma_k^2 + ||w||^2/\rho^2 \leq r^{-1}(r - 1) + 1 = 1$$

hence $v_{-i} \in B$, the unit ball of dimension $\sum_{k \neq i} m_k + n$. Consequently, $B_{-i}v_{-i} \in \{x : x'[B_{-i}B'_{-i}]^{-1}x \leq 1\}$ ($B_{-i}B'_{-i}$ is non-singular since $B_{-i}$ has rank $n$). Therefore, the disturbance viewed by player $i$ is a vector $\tilde{x} = B_{-i}v_{-i} \in \mathcal{E}(P_i)$ with

$$P_i = B_{-i}B'_{-i} = r \sum_{k \neq i} B_kB_k'^2 + r\rho^2I, \quad i \in [1,r].$$

which illustrates the linear-affine dependence of $P_i$ upon the parameters $\gamma_i^2$.

In the present context by definition (1) an IE is a collection $\{(\Omega_i, U_i), i \in [1,r]\}$ such that $\Omega_i = \mathcal{E}(X_i) \oplus \mathcal{E}(P_i)$ is $\mathcal{E}(P_i)$-robust invariant for (20). The question of interest is to find bounds $\gamma_i, \eta_i$ for which a IE exists.

Using the results of Sec. (II-B), the invariance condition for player $i$ under input-output constraints are expressed by

$$\begin{bmatrix} X_i \\ Y'_iB'_i + X_{a,i}A' \end{bmatrix} > 0$$

$$\begin{bmatrix} \gamma_i^2I \\ Y'_i \\ X_{a,i} \end{bmatrix} > 0$$

$$\begin{bmatrix} \eta_i^2I \\ X_{a,i}C' \end{bmatrix} > 0$$

$$X_i > 0$$

where

$$X_{a,i} = (1 + \alpha)X_i + (1 + \alpha^{-1})P_i. \quad (26)$$
and
\[ u_i = K_i x, \quad K_i = Y_i [X_{\alpha,i}]^{-1}. \]

We summarize the result in Theorem 14:

**Theorem 14:** If \((\eta^2, \gamma^2, Y, X)\) belong to the feasible region \(\mathcal{F}\) of (22-25) and \(\mathcal{F} \neq \emptyset\), then \((\Omega_i, U_i)\) is an IE with
\[
\Omega_i = \{ x \in \mathbb{R}^n : x' X_{\alpha,i} x \leq 1 \} \oplus \{ x \in \mathbb{R}^n : x' P_i x \leq 1 \}
\]
\[
U_i = \{ u_i \in \mathbb{R}^{m_i} : ||u_i||^2 = ||K_i x||^2 < \gamma_i^2 \}.
\]

Notice that invariant equilibria of ellipsoidal games admit linear strategies that can be parametrized in the convex parameter space \(\mathcal{F}\). This generalizes to decentralized control results known for centralized control. Thm 14 establishes a mapping of \(\mathcal{F}\) into \(E\), the set of IE, with the following properties

**Corollary 15:** Let \(\Pi : \mathcal{F} \mapsto E\) be the mapping established in Thm 14

i. \(\mathcal{F}\) is convex
ii. \((\eta^2, \gamma^2, Y, X) \in \mathcal{F} \Rightarrow (s \eta^2, t \gamma^2, t Y, t X) \in \mathcal{F}, \quad \forall s \geq t > 1\)
iii. \(\Pi(\eta^2, \gamma^2, Y, X) \subset \Pi(s \eta^2, t \gamma^2, t Y, t X), \quad \forall s \geq t > 1\)

**Proof:** Convexity follows because variables in \(\mathcal{F}\) are defined by LMIs. Point ii follows because:
(a) \(X_{\alpha,i}\) is increasing linear-affine in \(\gamma^2\), as per (21, 26); (b) LMIs (22-25) are monotonic-increasing in \(\gamma^2, Y, X\) for fixed \(\eta^2\); monotonic-increasing in \(\eta^2\) for fixed \(\gamma^2, Y, X\); (c) \(s \geq t\) ensures output constraints satisfied over the RI set for \(t = 1\) keep being satisfied over \(t > 1\). This also shows that an IE satisfies \(E \subset t E\) for \(t > 1\) provided \(s \geq t\), which is point iii.

IV. STRATEGY SELECTION

As discussed in the introduction, we distinguish the case in which sets \(U_i\) are known and common knowledge; and the case in which they are unknown.

A. known constraints

Assume sets \(U_i = \{ u : ||u||^2 < \gamma_i^2 \} \) are known and that LMIs (22-25) are feasible, e.g. \((\eta^2, \gamma^2, X, Y) \in \mathcal{F}\). Let \(S\) be the set of all such \(\eta^2\).

**Theorem 16:** There exists a unique point \(\hat{\eta}_i^2\) such that
\[
\hat{\eta}_i^2 = \inf_{\eta_i^2 \in S} \eta_i^2.
\]

**Proof:** By inspection, LMIs (22-25) are only coupled by the disturbance term (21). For fixed \(\gamma_i^2\) they decouple into a set of independent LMIs for each \(i\). Since the minimand is linear over strictly

---

1. We find expedient in the following to suppress subscript \(i\) and use notation \(\eta^2\) for a vector with components \(\eta^2_1 \ldots \eta^2_r\).

2. For fixed \(\alpha\). One could select one individual \(\alpha_i\) for each LMI. Increasing the degrees of freedom of the parametrization also increases the chances to get a solution, if one exists. However the search of \(\alpha\) would no longer be one-dimensional (see Remark 12) and this complicates the analysis.
convex constraints, there exist a unique $\eta_i^2, \bar{X}, \bar{Y}$ such that $\bar{\eta}_i^2 = \min_{\bar{F}} \eta_i^2$ where $\bar{F}$ is the closure of $F$. Since $\bar{\eta}^2 \in \bar{S}$ we have $\bar{\eta}^2 = \bar{\eta}_i^2$.

As it is in the interest of player $i$ to have $||y_i||^2 < \eta_i^2$ we conclude that there is a non-cooperative and optimal way to select a point $\eta_i^2 \in S$. Moreover, the data needed to evaluate $\inf \eta_i^2$ subject to (22-25) are all known to player $i$ and $\gamma^2$ is common knowledge. Hence each player can compute $\Omega = \cap_i \mathcal{E}(\bar{X}_{\alpha,i})$ and conclude that if other players adopt equilibrium strategies, then he can guarantee $\sup_{\Omega} \eta_i^2$ (again, a convex problem). If on the other hand one or more players deviate, then he can only guarantee $\hat{\eta}_i^2$.

B. unknown constraints

If $U_i$ are not known, $\gamma^2$ becomes a decision variable and a difficulty arises due to the multiplicity of IE and the coupling introduced by the variables $\gamma^2$ in the LMIs through (21). The possibility of a non-cooperative selection of an IE is ruled out by the following result. Let $F_{\gamma}$ be the set of $(\eta^2, X, Y)$ such that $(\eta^2, \gamma^2, X, Y) \in F$ and let $\bar{F}_\gamma$ be its closure. Consider the functions

$$J_i : \mathbb{R} \times \mathbb{R}^{r+1} \mapsto \mathbb{R}, \quad J_i(\gamma_i^2, \gamma_{-i}^2) = \min_{\bar{F}_\gamma} \eta_i^2.$$

Being the minimum of a convex function on a closed convex set, $J_i$ is well defined and by Lemma (13) $\arg\min_{\gamma_i^2} J_i(\gamma_i^2, \gamma_{-i}^2) = \hat{\gamma}_i^2$ so there is no point in $F$ satisfying the non-cooperative (Nash) condition

$$\hat{\gamma}_i^2 = \arg\min_{\gamma_i^2} J_i(\gamma_i^2, \gamma_{-i}^2)$$

showing that cooperation is necessary. The question is whether cooperative communication is strictly required for the selection of a specific IE; or, whether it is possible to select an IE purely on the basis of collective rationality. We argue that the latter is the case and that, in the present decentralized information context, the Nash bargaining solution is the natural way to provide such a rationality.

C. Nash bargaining solution

We recall that if $S \in \mathbb{R}^r$ is the bargaining domain of a $r$-person game, a disagreement point is a point in $\mathbb{R}^r$ whose $i$-th coordinate denotes an ideal utility level for player $i$ that cannot be collectively enforced: $d \notin S$. The Nash bargaining solution [8], [12] prescribes $s = \arg\max_S (d_1 - s_1) \times \ldots \times (s_r - d_r)$. It is well known that $s$ is a Pareto-optimal solution if $S$ is closed convex. In our context, $S$ is represented by the set

$$S = \{ \eta^2 : (\eta^2, \gamma^2, X, Y) \in F \}.$$

Although this is an unbounded set, there is no loss in generality in assuming each $\eta_i^2$ to be bounded above by $\bar{s} = \max_{\bar{S}}$ (with max meant componentwise) where

$$\bar{s}_k = \arg\min_{\bar{S}} \eta_k^2$$

where $\bar{S}$ is the closure of $S$. Consequently the Nash bargaining solution can be characterized as

$$s = \arg\max_{\bar{S}} (\bar{s} - s_1) \times \ldots (\bar{s} - s_r).$$

Notice that the maximand is a concave function defined on a convex set, hence quite amenable to LMI optimization methods. Furthermore, being $\bar{S}$ convex, $s$ is Pareto-optimal.

3Technically, we have a single Pareto-optimal point, which is also a non-cooperative equilibrium in dominating strategies: little left to be desired!
D. theoretical justification

When $U_i$ are unknown, each player should strive to select a IE and the corresponding equilibrium strategy by using common knowledge of $A, B_i, C_i,$ and $\rho$ and the rationality of the opponents. This can be accomplished on the basis of the following considerations

- since a non-cooperative solution does not exist, any solution $\eta^2 \in \bar{S}$ which is not Pareto-optimal cannot be chosen as rational
- $B_i = B$ and $C_i = C$ for all $i$ then, for reasons of symmetry, the solution should satisfy $\eta_i^2 = \gamma_i^2, \gamma_i^2 = \gamma^2, X_i = X, Y_i = Y$.
- since $\mathcal{F}$ is the domain of a LMI system which is homogeneous in its variables, if $\eta^2$ is the solution when feasible parameters are $\gamma^2, X, Y, \rho$ then $t\eta^2$ should also be a solution when feasible parameters are $t\gamma^2, tX, tY, t\rho$
- the solution should be robust to parameter changes in the following sense: if $\eta^2$ is the solution when system parameters are $A, B_i, C_i, \gamma, \rho$ it should not change when perturbed parameters $\tilde{A}, \tilde{B}_i, \tilde{C}_i, \tilde{\gamma}, \tilde{\rho}$ are such as to produce a feasible region $\tilde{\mathcal{F}}$ sharing with $\mathcal{F}$ a neighborhood of $\eta^2$.

In cooperative Game theory these four properties are recognized as Pareto-optimality, symmetry, scale invariance and contraction independence. It is well known that the Nash bargaining solution is the only cooperative solution concept satisfying the above properties and for this reason it appears quite fit to the present context.

V. APPLICATION IN FINANCE

Consider a stock market in which two sets of agents operate: a large number of small investors and a small number of fund managers. Assume there are $r$ fund managers. Manager $i$ trades in a number $m_i$ of funds, $B_{i,1} \ldots B_{i,m_i}$ and each fund, say manager $i$’s $k$-th fund, is a portfolio made up of shares

$$
\begin{bmatrix}
  b_{i,k,1} \\
  \vdots \\
  b_{i,k,n}
\end{bmatrix} = B_{i,k}
$$

of $n$ underlying stocks. Let $x_1 \ldots x_n$ be the one-period returns of the $n$ underlying stocks. Let $B \in \mathbb{R}^{n \times m}$ collect the funds of all managers ($m = \sum_i m_i$)

$$
B = [B_1 \mid B_2 \mid \ldots \mid B_r], \quad B_i \in \mathbb{R}^{n \times m_i}.
$$

As a buyer, fund manager $i$ contributes to the demand of stocks by $B_iu_i$. As a seller, manager $i$ supplies a set of funds yielding one-period returns $y_i = B'_ix$. The total traded volume is $B_{i,1}u_1 + \cdots + B_{r,u_r} + w$ with $w$ representing the volume traded by the small traders. Assuming stock returns react linearly to traded volumes, the market is described by

$$
x^+ = Ax + B_{i,1}u_1 + \cdots + B_{r,u_r} + w \\
y_1 = B'_ix \\
\vdots \\
y_r = B'_rx.
$$

where $A$ is a dynamic matrix, representing correlations between stocks. In absence of trade, stock returns should tend to zero, so it is reasonable to assume $A$ to be a Schur matrix. Market tends to
be perfect (or competitive) when the size of $w$ is so large as to make the effect of $Bu$ negligible; imperfect (or oligopolistic) when $Bu$ affects $x$ significantly; in which case, speculative opportunities for fund managers (=strategic behaviour) open up. Assume managers face symmetric constraints on the units traded $±u \in U_i$. Then, normalizing to zero the benchmark of trader $i$, the tracking objective of manager $i$ can be expressed as $y_i \in Y_i$. One of the attractive features of investment contracts, which has gained popularity in the last decade or so, is the possibility of guaranteeing, for the length of the contract, benchmark deviations smaller than $\eta^2$, conditional to exogenous shocks $w$ not exceeding in the period a given threshold, $||w||^2 < \rho^2$. Among the questions of interest in this setting is what volume of trade would be necessary to make that contract possible. If each trader faces a constraint $||Bu_i|| \leq M_i$ on the maximum volume he is able to trade, we interpret it $||u_i||^2 < \gamma^2 \leq M^2_i/||B_i||^2$ and are within the model-framework discussed in this paper: the existence of an invariant equilibrium can provide an answer. An interesting aspect in this framework is that, under a technical assumption on the number of funds, for bounded $\eta^2$ the volume traded by fund managers must be large enough compared to the volume traded by small investors in order for an IE to exist.

**Assumption 17:** $B$ has rank $m$ and

$$m \geq \begin{cases} \frac{n}{2} & \text{if } n \text{ is even}, \\ \frac{n+1}{2} & \text{if } n \text{ is odd}. \end{cases}$$

The assumption can be viewed as a weak form of market completeness.

**Theorem 18:** Under (17) an ellipsoidal IE exists if and only if

$$\sup \frac{||w||}{||Bu||} \leq \hat{\rho}.$$

**Proof:** From Corollary (15) the set of IE is an unbounded convex domain $\mathcal{F}$ in the variables $(\gamma^2_i, \eta^2_i, X_i, Y_i)$ and equilibrium strategies are $u_i = Y_i[X_{\alpha,i}]^{-1}x$ where $X_{\alpha,i}$ is a linear function of $X_i, \gamma^2, \rho^2$, see (21,26).

If $||Bu|| < \gamma$ and $||y|| < \eta$ over $\mathcal{E}(X_{\alpha,i})$, the vector $z \in \mathbb{R}^{n+m}$

$$z = \begin{bmatrix} BK \\ B' \end{bmatrix} x$$

with $K = Y_i[X_{\alpha,i}]^{-1}$ is bounded for all $x \in \mathcal{E}(X_{\alpha,i})$. Notice that, after elimination of redundant strategies, we can take $\text{rank}(K) = m$ wlog. Since $\text{rank}(B) = m$, for any full-rank $K \in \mathbb{R}^{m \times n}$ matrix

$$M = \begin{bmatrix} BK \\ B' \end{bmatrix} \in \mathbb{R}^{(n+m)\times n}$$

has rank $2m$. From Assumption (17) it follows $\text{rank}(M) = n$. Hence boundedness of $z$ implies the same for all $x \in \mathcal{E}(X_{\alpha,i})$ and $\mathcal{F}$ is bounded. Therefore there exists $\hat{\rho}$ such that

$$\max_{\mathcal{F}} \rho^2 = \hat{\rho}^2 \gamma^2.$$

Conversely, if such a $\hat{\rho}$ exists, $\mathcal{E}(X_{\alpha,i})$ is a bounded ellipsoid and $Bu$ and $y$ are bounded. ■

In conclusion, when market power of fund-traders is such as to overcome significantly that of other traders, guaranteed benchmark tracking is strongly dependent on the existence of an IE. In this sense, IE could be regarded as an innovative element in the theory of market imperfection.
VI. Appendix: Implementation and Coding

The three LMIs (22-25) can be implemented (in MATLAB or other language) by defining, for ease of coding

\[ m = \sum_{i=1}^{r} m_i, \quad p = \sum_{i=1}^{r} p_i \]

\[ G_i = \text{diag}(M_i \gamma^2) \in \mathbb{R}^{m \times m}, \quad F_i = \text{diag}(N_i \eta^2) \in \mathbb{R}^{p \times p}. \]

where \( M_i \in \mathbb{R}^{m \times r}, N_i \in \mathbb{R}^{p \times r} \)

\[
M_i = \begin{bmatrix}
1_{m_1} & 0 & \ldots & 0 \\
0 & 1_{m_2} & \ldots & 0 \\
0 & \ldots & \ldots & 0 \\
0 & 0 & \ldots & 1_{m_r}
\end{bmatrix}, \quad 1_{m_i} = 0; \quad \gamma = \begin{bmatrix}
\gamma_1^2 \\
\gamma_2^2 \\
\vdots \\
\gamma_r^2
\end{bmatrix}
\]

\[
N_i = \begin{bmatrix}
1_{p_1} & 0 & \ldots & 0 \\
0 & 1_{p_2} & \ldots & 0 \\
0 & \ldots & \ldots & 0 \\
0 & 0 & \ldots & 1_{p_r}
\end{bmatrix}, \quad 1_{p_i} = 0; \quad \eta = \begin{bmatrix}
\eta_1^2 \\
\eta_2^2 \\
\vdots \\
\eta_r^2
\end{bmatrix}
\]

and

\[ M_{-i} = I_m - M_i, \quad N_{-i} = I_p - N_i. \]

It is immediate to verify

\[ P_i = rBG_iB' + r\rho^2I \]

\[ \gamma_i^2 I_{m_i} = \text{diag}(M_{-i} \gamma^2) = \Gamma_i \]

\[ \eta_i^2 I_{p_i} = \text{diag}(N_{-i} \eta^2) = E_i. \]

The ”decision varables” in the LMI system are \( \gamma^2, \eta^2 \) and \( X_i, Y_i \), for \( i = 1 \ldots r \). With the above positions, the LMI system becomes

\[
\begin{bmatrix}
X_i (1 + \alpha)AX_i + (1 + \alpha^{-1})A(rBG_iB' + r\rho^2I) + B_iY_i \\
* & (1 + \alpha)X_i + (1 + \alpha^{-1})(rBG_iB' + r\rho^2I)
\end{bmatrix} > 0
\]

\[
\begin{bmatrix}
\Gamma_i & Y_i \\
* & (1 + \alpha)X_i + (1 + \alpha^{-1})(rBG_iB' + r\rho^2I)
\end{bmatrix} > 0
\]

\[
\begin{bmatrix}
E_i (1 + \alpha)C_iX_i + (1 + \alpha^{-1})C_i(rBG_iB' + r\rho^2I) \\
* & (1 + \alpha)X_i + (1 + \alpha^{-1})(rBG_iB' + r\rho^2I)
\end{bmatrix} > 0
\]

for \( i = 1 \ldots r \) (a star indicates symmetric term for short). Computation of Nash bargaining solution turns out to be quite expedient even in relatively large dimensions. For a 4 player game with state dimension \( n = 10 \) a solution is found in 96 iterations for a total CPU time of 40.6 secs.\(^4\)

\(^4\)using \texttt{cvx} [2]. I am grateful to M. Grant, S. Boyd, and Yinyu Ye for their contribution to the community of LMI users with an excellent freeware product.
REFERENCES