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Numerical study of the homicidal chauffeur game

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Abstract

The Isaacs' homicidal chauffeur differential game is considered. In this game a chauffeur (the pursuer P) minimizes the capture time of a pedestrian (the evader E). The objective of the pedestrian is to avoid the capture or to maximize the capture time. The capture occurs when the distance between the players becomes less or equal than a given value (the capture radius). The pursuer's control is the rate of turn, the evader governs choosing directions of his velocity. The velocity of the pursuer is greater than that one for the evader but his maneuverability is bounded. Numerical solutions to this problem are obtained using an algorithm proposed by the authors for computing level sets of the value function.

Keywords: *Differential games, time-optimal control, homicidal chauffeur game*

Introduction

The homicidal chauffeur game was formulated by R.Isaacs more than thirty years ago [1]. Since that time, many authors studied this problem in various ways. The most complete qualitative solution was given in the PhD dissertation of A.Merz [2]. In this work, the region of parameters of the problem was divided into certain subregions. In each subregion, the structure of the solution was described.

Very often (see [3–5]) the dynamics of the homicidal chauffeur game was used, but the statement of the problem differed from the one of Isaacs. In [3], for example, a surveillance-evasion differential game of degree with the pursuer's detection zone in the shape of a circle was considered.

Many papers on differential games are devoted to the development of algorithms for solving nonlinear differential games in the plane [6–10]. In some of these papers (see, i.e., [10]), the homicidal chauffeur game is used to demonstrate the efficiency of the algorithms.

In this paper, the homicidal chauffeur game is investigated using the algorithm proposed by the authors for computing level sets of the value function. Our methods are based on the general theory of differential games [11, 12]. The algorithm is a natural extension of the algorithms from [13], and exploits ideas of the algorithms [14–16] for linear time-optimal differential games in the plane. Some experience [13, 14, 17, 18] in solving differential games of kind [1] in the plane helps to found out very complicated types of solutions and to verify the solution's validity. The computation results are consistent with those obtained by A.Merz.

It is the well-known fact due to works of R.Isaacs, J.V.Breakwell, and A.Merz that the solution of the homicidal chauffeur game may contain various types of singularities inherent to differential games: switching, dispersal, equivocal, universal, and focal lines.

The our algorithm's peculiarity consisting in the use of reverse time extremal trajectories makes possible to recognize types of singular lines.

Statement of the problem

The dynamics of the homicidal chauffeur game in reduced coordinates has the form

$$\begin{aligned} \dot{x}_1 &= -\tfrac{w^{(1)}}{R} x_2\ \varphi + v_1 \\ \dot{x}_2 &= \ \ \tfrac{w^{(1)}}{R} x_1\ \varphi + v_2 - w^{(1)}, \qquad \mid \varphi \mid \leq 1, \quad v \in Q. \end{aligned} \tag{1}$$

Here $(x_1,\ x_2)'$ is the state vector, $w^{(1)}$ and R are constants which have the sense of the pursuer's velocity and the minimal radius of turn, respectively. The objective of the control φ of the first player is to minimize the time of attaining a given terminal set M. The objective of the control $v = (v_1, v_2)'$ of the second player is to maximize this time. So, the payoff of the game is the time of attaining the terminal set.

Let $\theta \geq 0$. The level set (the Lebesgue set) of the value function is denoted by $W(\theta, M)$. This is the set of all points in the plane such that the first player can guarantee the transition of trajectories of the system (1) to the terminal set M within the time θ. In this paper, the basic idea of the algorithm for computing sets $W(\theta, M)$ is described. The classical formulation of the homicidal chauffeur game assumes that the sets M and Q are circles of the radii l (capture radius) and $w^{(2)}$, respectively, with the centers at the origin. It is assumed that $w^{(1)} > w^{(2)}$. With the algorithm proposed, the sets $W(\theta, M)$ can be computed for arbitrary sets M and Q in the plane. This enable us to obtain new types of solutions and to study some interesting cases.

The algorithm

The algorithm is based on ideas of the algorithms for linear time-optimal game problems [14–16]. The set $W(\theta, M)$ is formed via a step-by-step backward procedure giving a sequence of embedded sets

$$W(\Delta, M) \subset W(2\Delta, M) \subset W(3\Delta, M) \subset ... \subset W(i\Delta, M) \subset ... \subset W(\theta, M). \tag{2}$$

Here Δ is the step of the backward procedure. Each set $W(i\Delta, M)$ consists of all initial points such that the first player brings system (1) into the set $W((i-1)\Delta, M)$ within the time duration Δ. We put $W(0, M) = M$.

The crucial point of the algorithm is the computation of "fronts". The front F_i (Figure 1) is the set of all points of $\partial W(i\Delta, M)$ for which the minimum guaranteeing time of the achievement of $W((i-1)\Delta, M)$ is equal to Δ. For other points of $\partial W(i\Delta, M)$ the optimal time is less than Δ. The line $\partial W(i\Delta, M) \setminus F_i$ possesses the properties of the barrier [1]. The front F_i is designed using the previous front F_{i-1}. For the first step of the backward procedure, F_0 coincides with the usable part [1] Γ_0 of the boundary of M.

Let us explain how the fronts can be constructed. Denote $p(x) = (-x_2, x_1)' \cdot w^{(1)}/R$, $g = (0, -w^{(1)})'$. Using this notation, the equations (1) can be rewritten as follows: $\dot{x} = p(x)\varphi + v + g$. Suppose the front F_{i-1} is a smooth curve. Let x_* be an arbitrary point of F_{i-1}, and ℓ is the normal vector to the front at x_*. Let $\varphi^\circ = \arg \min\limits_{|\varphi| \leq 1} \ell' p(x_*)\varphi$, $v^\circ = \arg \max\limits_{v \in Q} \ell' v$. We call φ°, v° the extremal controls. The controls φ° and v° are chosen from the conditions of minimizing and, respectively, maximizing the projection of the velocity vector of (1) onto the direction ℓ. If the vector x_* is collinear to ℓ, then any control $\varphi \in [-1, 1]$ is extremal. If Q is a polygon in the plane, and ℓ is collinear to some normal vector to an edge $[q_1, q_2]$ of Q, then any control $q \in [q_1, q_2]$ is extremal.

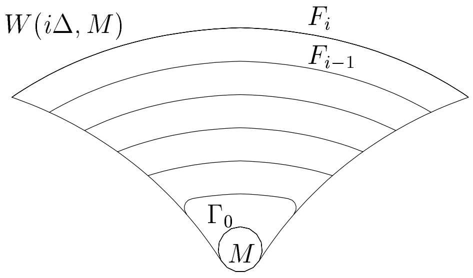

Figure 1. Construction of the sets $W(i\Delta, M)$.

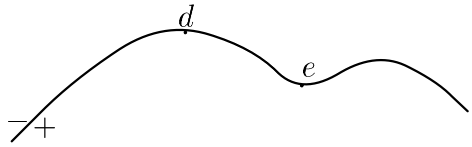

Figure 2. Local convexity and concavity.

After computing the extremal controls, the extremal trajectories issued from the front's points in the reverse time are considered: $x(\tau) = x_* - \tau\ (p(x_*)\varphi^\circ + v^\circ + g)$. The ends of these trajectories at $\tau = \Delta$ are used to form the next front F_i. If the extremal control φ° is not unique at some point $x_* \in F_{i-1}$, then the the segment $\Phi(x_*) = \{ \bigcup\limits_{\varphi^\circ \in [-1,1]} (x_* - \Delta\ (p(x_*)\varphi^\circ + v^\circ + g))\}$ is considered instead of the single point. Similarly, if the extremal control v° is not unique, the segment $\Xi(x_*) = \{ \bigcup\limits_{v^\circ \in [q_1, q_2]} (x_* - \Delta\ (p(x_*)\varphi^\circ + v^\circ + g))\}$ is considered.

For each front, we distinguish points of the local convexity and points of the local concavity. In Figure 2, d is a point of the local convexity, and e is a point of the local concavity. If x_* is a point of the local convexity and the extremal control φ° is not unique, we obtain a local picture like the one shown in Figure 3A after issuing the extremal trajectories from the point x_*. Here, the additional segment $\Phi(x_*)$ is appeared on the new front F_i. If the extremal control v° is not unique, we obtain a local picture similar to the one shown in Figure 3B: the "swallow tail " $\beta_1 \xi \beta_2$ does not belong to the new front F_i and it is taken away. For points of the local concavity, there is an inverse situation: if the extremal control φ° is not unique, a swallow tail that should be removed appears; if the extremal control v° is not unique, an additional segment $\Xi(x_*)$ appears on the new front F_i. If both φ° and v° are non-unique, the insertion or the swallow tail arises depending on which of segments $\Phi(x_*)$ or $\Xi(x_*)$ is greater.

In the course of numerical computations, we operate with polygonal lines instead of smooth curves. Two normal vectors to the links $[a, b], [b, c]$ of the polygonal line are considered at each vertex b (Figure 4). The algorithm treats all possible variants of disposition of the vectors $\ell_{[ab]}, \ell_{[bc]}$, normals to the edges of Q, and the vector b. In Figure 4, for instance, the case is shown where the vector b is between vectors $\ell_{[ab]}$, $\ell_{[bc]}$, and the normals n_1, n_2 to the set Q are between the vectors b and $\ell_{[bc]}$. The extremal controls of

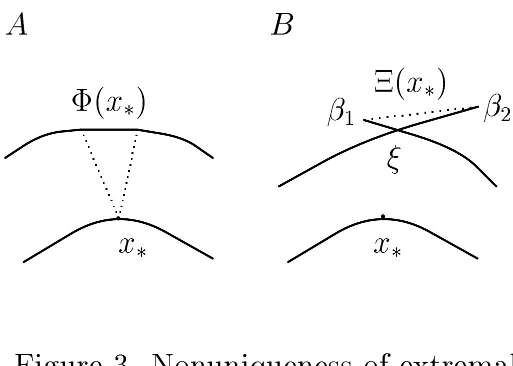

Figure 3. Nonuniqueness of extremal controls in the case of local convexity.

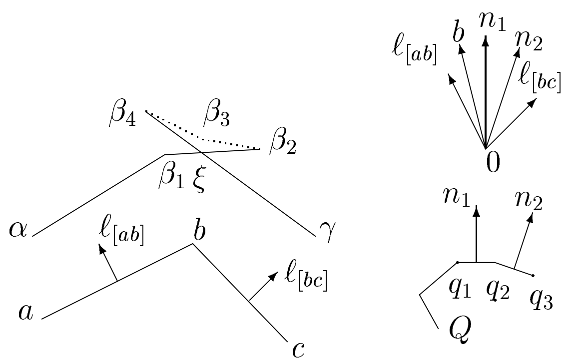

Figure 4. Example of local constructions.

the players are computed for each of these vectors, and the extremal trajectories are issued from the points a, b, c. The ends of these trajectories computed at $\tau = \Delta$ give a local picture depicted in Figure 4. In the case considered, four extremal trajectories were issued from the point b. Their ends are: $\beta_1, \beta_2, \beta_3, \beta_4$. The segment $[\beta_1, \beta_2]$ appears due to nonuniquiness of the extremal control φ° for the vector b. The segments $[\beta_2, \beta_3]$ and $[\beta_3, \beta_4]$ arise due to nonuniquiness of the extremal control v° for the vectors n_1 and n_2. After removing the swallow tail $\xi\beta_4\beta_3\beta_3\beta_2\xi$, the polygonal line $\alpha\beta_1\xi\gamma$ is obtained to be a fragment of the next front.

More detailed description of such local constructions is given in [16] The main difference from the case of the linear dynamics [16] is that the extremal control of the first player can change its value not only at front's vertices but also at some interior points of front's links. In the game considered, such a switching may occur only once for each front's link.

Since the algorithm is based on the computation of the extremal trajectories, the singular lines can be obtained (see [16] for the idea).

Examples

In this section, the results of computing the sets $W(\tau, M)$, $\tau = i\Delta$, are discussed.

In all examples presented, the following values of parameters of the problem are used: $w^{(1)} = 0.6$, $w^{(2)} = 0.2$, $R = 2$. The set Q is a 25-polygon inscribed into the circle of radius $w^{(2)}$ with the center at $(0, 0)$. The set M is a 15-polygon inscribed into the circle of radius 0.015. The center of the circle is in $(0, -0.23)$, $(0, 0.2)$, $(0, 0.5)$, $(0, -0.45)$, $(0.35, -0.3)$, $(0.2, 0.3)$ for Figures 5A–F, respectively. The step Δ is 0.0025.

In Figure 5A, the sets $W(\tau, M), \tau = 10\Delta i$, are depicted. The computations are done up to $\tau = 0.95$ when the self-intersection of the front happens. The point of the self-intersection divides the front into two parts: the interior part (adjoining to the gap) and the exterior one (enclosing the whole consruction). For $\tau > 0.95$, the computations should be carried out from the exterior and interior parts of the last front independently.

In Figure 5B (5C) the sets $W(\tau, M), \tau = 11\Delta i$ $(\tau = 12\Delta i)$, are depicted. The self-intersection of the front happens at $\tau = 0.7$ $(\tau = 0.6)$. The computations are continued from the interior part of the self-intersecting front. The gap between this part of the front and the barrier lines is completly filled out. The computations from the exterior part of the self-intersecting front were not carried out. In Figure 5B, the last interior front corresponds to $\tau = 1.045$. In Figure 5C, the second self-intersection of the front happens at $\tau = 1.045$ in the process of filling out the gap. The closed front arises. The computations from this closed front are done until it shrinks into a point at $\tau = 1.125$.

In Figure 5D, the sets $W(\tau, M), \tau = 8\Delta i$, are shown. The self-intersecting front, which corresponds to $\tau = 0.355$, is drawn with the thick dash line. The computations are continued from both interior and exterior parts of the front. The last interior (exterior) front corresponds to the time $\tau = 0.38$ $(\tau = 0.76)$. The sets $W(\tau, M)$ for $0.355 < \tau < 0.38$ are doubly connected.

In Figure 5E, the sets $W(\tau, M), \tau = 13\Delta i$, are shown. The gap between the interior part of the self-intersecting front and the barrier lines is completly filled out as it is in Figure 5B. But interior fronts have much more complicated structure.

The most complicated case is presented in Figure 5F. The sets $W(\tau, M), \tau = 10\Delta i$, are depicted. The self-intersection of the interior front, which corresponds to $\tau = 0.9$ and is drawn with the thick dash line, produces two gaps. The next front consists of three parts: an exterior part (which is not shown), and two interior parts (two loops inside the dash contour). As the result, the sets $W(\tau, M)$ for $0.9 < \tau < 0.95$ are triple connected.

For all Figures 5A–F, barriers are drawn with thick lines.

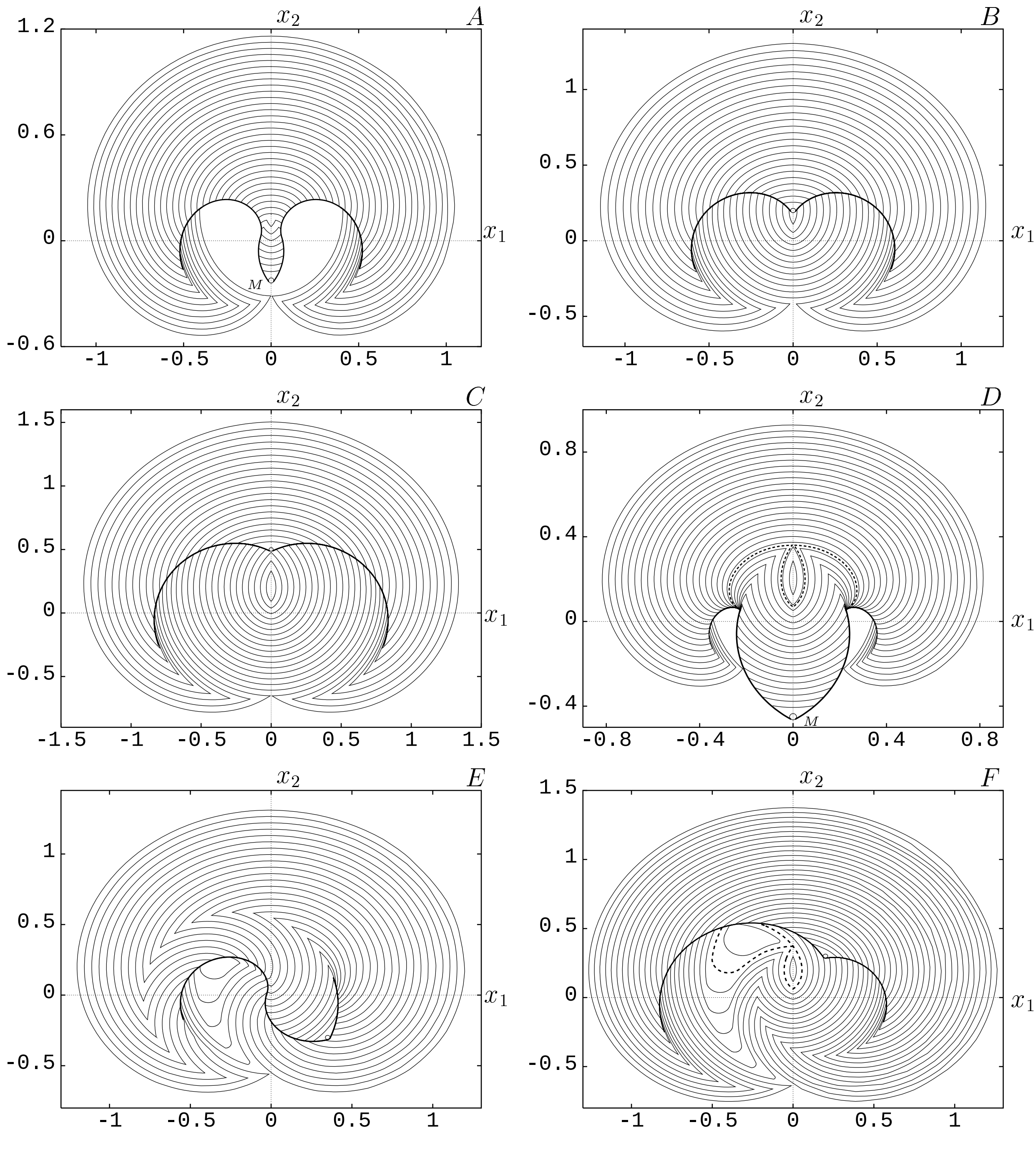

Figure 5. Examples of numerically computed sets $W(i\Delta, M)$.

Semipermeable curves in the homicidal chauffeur game

In [13, 14, 17, 18], algorithms for solving linear and some nonlinear differential games of kind are described. These algorithms are based on the computation of the semipermeable curves [1].

In this section, the results of some analysis of the semipermeable curves in the homicidal chauffeur game will be given. Using these results, one can find the solvability sets of the game of kind for any terminal set M and an arbitrary set Q. Since the set $W(\theta, M)$ converges to the solvability set of the corresponding game of kind as $\theta \to \infty$, solutions to the game of kind can be used for the verification of the computation of the sets $W(\theta, M)$.

Let us explain now what semipermeable curves mean. Let

$$H(\ell, x) = \min_{\varphi} \max_{v} \ell' f(x, \varphi, v) = \max_{v} \min_{\varphi} \ell' f(x, \varphi, v), \quad x \in R^2, \ell \in R^2,$$

be the Hamiltonian of the game. Here $f(x, \varphi, v) = p(x)\varphi + v + g$. Fix $x \in R^2$ and consider ℓ such that $H(\ell, x) = 0$. Denote $\varphi^* = \arg\min_{\varphi} \ell' f(x, \varphi, v)$, $v^* = \arg\max_{v} \ell' f(x, \varphi, v)$. It holds: $\ell' f(x, \varphi^*, v) \leq 0$ for any $v \in Q$ and $\ell' f(x, \varphi, v^*) \geq 0$ for any $\varphi \in [-1, 1]$. This means that the direction $f(x, \varphi^*, v^*)$ which is orthogonal to ℓ separates the vectograms $\Phi(v^*) = \bigcup_{\varphi \in [-1,1]} f(x, \varphi, v^*)$ and $\Xi(\varphi^*) = \bigcup_{v \in Q} f(x, \varphi^*, v)$ of the first and second players (Figure 6). Such a direction is called semipermeable.

So, the semipermeable directions are defined by the roots of the equation $H(\ell, x) = 0$. We will distinguish the roots "$-$" to "$+$" and the roots "$+$" to "$-$". When defining these roots, we will suppose that $\ell \in E$ where E is a closed polygonal line around the origin. We say that ℓ_* is a root "$-$" to "$+$" if $H(\ell_*, x) = 0$ and $H(\ell, x) < 0$ $(H(\ell, x) > 0)$ for $\ell < \ell_*$ $(\ell > \ell_*)$ that are sufficiently close to ℓ_*. The notation $\ell < \ell_*$ means that the direction of the vector ℓ can be obtained from the direction of the vector ℓ_* using the counterclockwise rotation by the angle not exceeding π. The roots "$-$" to "$+$" and the roots "$+$" to "$-$" are called the roots of the first and second type, respectively.

One can prove that, in the game considered, the equation $H(\ell, x) = 0$ has at least one root of the first type and one root of the second type. Moreover, it has two roots of the first type and two roots of the second type at most. We denote the roots of the first type by $\ell^{(1),i}(x)$ and the roots of the second type by $\ell^{(2),i}(x)$. One can find the domains of the functions $\ell^{(j),i}(\cdot)$, $j = 1, 2$, $i = 1, 2$. The form of these domains is shown in Figure 7.

It can be proved that the function $\ell^{(j),i}(\cdot)$ satisfies the Lipschitz condition in any closed subset of its domain. So, we can consider the following differential equation

$$dz/dx = \Pi \ell^{(j),i}(x), \tag{3}$$

where Π is the matrix of rotation by the angle $\pi/2$ (the rotation's direction depends on j). Since the tangent at each point of phase trajectories of this equation is a semipermeable direction, the trajectories are semipermeable curves. It means that the first player can keep one side of the curve (say, positive side) and the second player can keep another side (negative side). So, the equation (3) specifies a family $\Lambda^{(j),i}$ of semipermeable curves. The families $\Lambda^{(j),i}$, $j = 1, 2$, $i = 1, 2$, are depicted in Figure 8.

The curves of different families belonging to the same type can be sewed in some cases so that the semipermeability property will be preserved. The procedure for computing the solvability set of the game of kind is based [17, 18] on the issuing two semipermeable

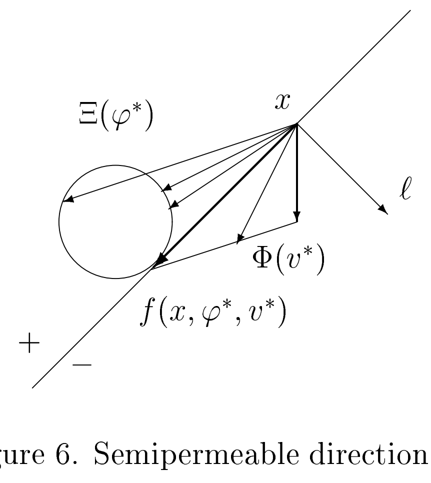

Figure 6. Semipermeable direction.

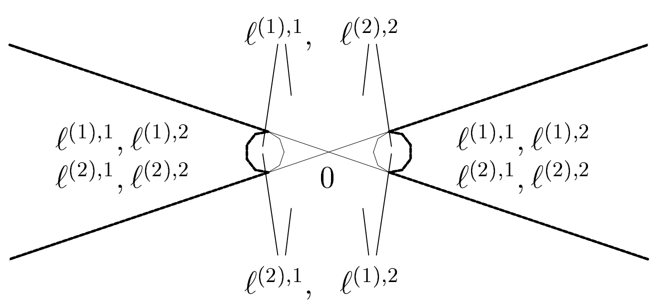

Figure 7. Domains of $\ell^{(j),i}$.

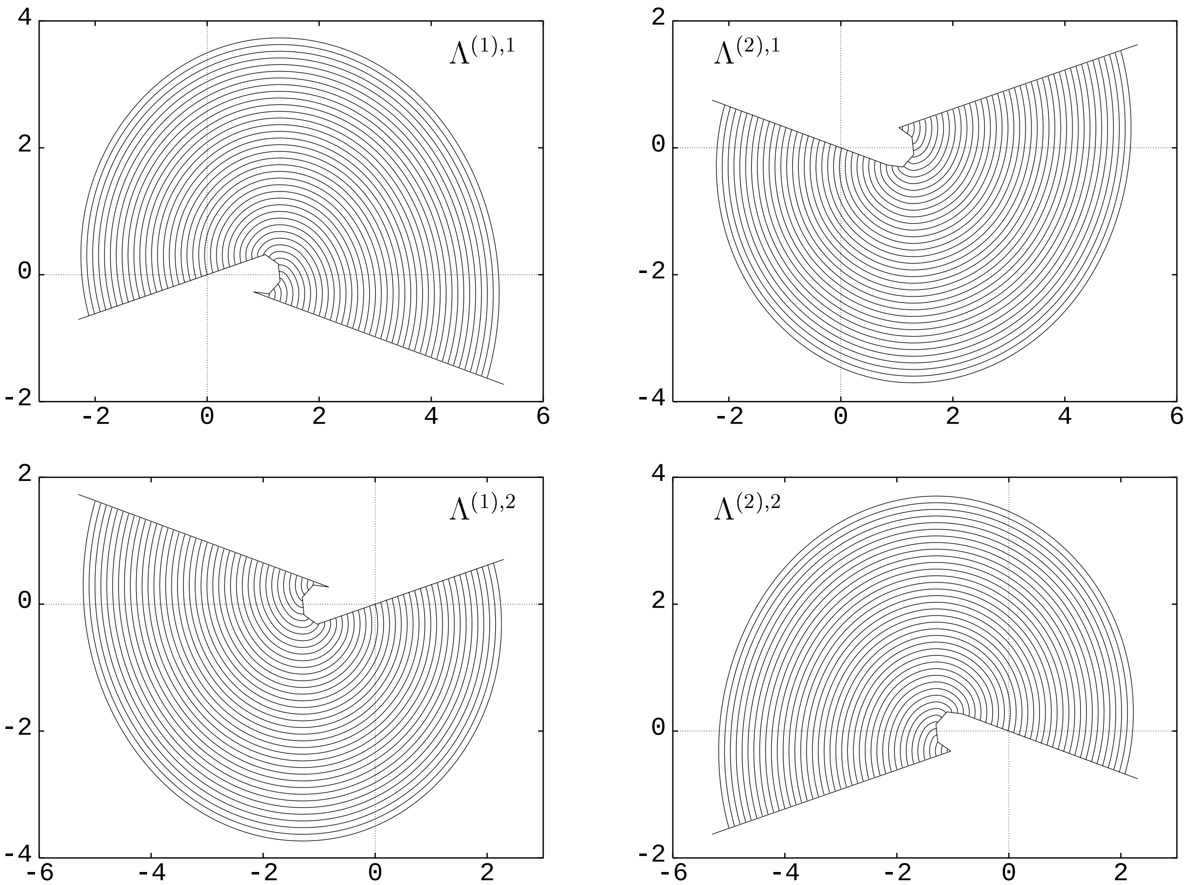

Figure 8. Families of semipermeable curves.

curves of the first and second type (they are faced each to other with positive sides) from the endpoints of M's usable part, on the sewing semipermeable curves of two different families belonging to the same type, and on the analysis of mutual dispositions of composite curves. Here, it will be shown with some example how this methodology can be used to check the validity of the computation of $W(i\Delta, M)$.

Let us consider the following example. The terminal set M is the regular 25-polygon inscribed into the circle of the radius 0.015 with the center at $(0.2, -0.4)$. The set Q is the triangle with vertices $(-1.2, 1), (1.2, 1), (0, -6)$; $w^{(1)}$=0.8. In this example, the solvability set of the game of kind is the whole plane. This can be proved using the following consideration of semipermeable curves. The semipermeable curves $p^{(2),1} \in \Lambda^{(2),1}$ and $p^{(1),2} \in \Lambda^{(1),2}$ issued from the endpoints of the usable part of M do not intersect each other before they finish at the boundaries of the corresponding domains (Figure 9A). The conjunction of $p^{(2),1}$ and $p^{(2),2}$ at the point s_1 is smooth. It provides the semipermeability property of $p^{(2),1} \bigcup p^{(2),2}$ at s_1. The composite curve $p^{(2),1} \bigcup p^{(2),2}$ does not intersect $p^{(1),2}$. Though the conjunction of the arc $rs_2 \in p^{(1),2}$ and and the curve $p^{(1),1}$ is non-smooth, the semipermeability property is fulfilled at the conjunction point s_2. The composite semipermeable curves of the first and the second types do not intersect each other. Further semipermeable curves are not being produced.

The sets $W(\tau, M)$ computed numerically are shown in Figure 9C. One can see that the curve $p^{(2),1} \bigcup p^{(2),2}$ is one of the two barriers. The curve $p^{(1),2}$ is the other barrier. The structure of the fronts near $p^{(1),1}$ is shown in Figure 9B. The fronts lie above the curve $p^{(1),1}$ even though they come very close to it. More detailed consideration of the behavior of fronts for sufficiently large τ shows that $p^{(1),1}$ is not a barrier.

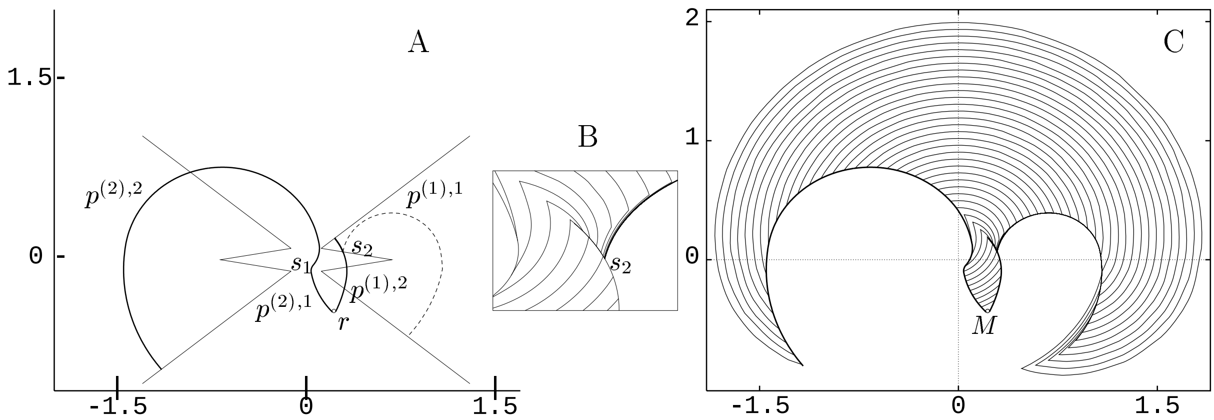

Figure 9. Example with the triangle set Q.

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