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Two Weak Pursuers in a Game Against a Single Evader

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Abstract—We consider an antagonistic differential game where the first player controls the actions of two pursuers that aim to minimize, at a given time instant their miss with respect to an evader. The second (maximizing) player is identified with the evader. We study the case when dynamic capabilities of pursuers are less than the capabilities of the evader. We propose a quasioptimal control method for the first player with switching lines. We also show modeling results.

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1. INTRODUCTION

The works [1–3] consider a model linear pursuit problem with two pursuers and one evader. Three inertial objects move along a straight line. Control for each of them is scalar and bounded in absolute value. At a predefined instant T_1, distance between the first pursuer and the evader is measured; at an instant T_2, the distance between the second pursuer and the evader. Pursuers are coordinated, and their objective is to minimize the payoff function which is the minimum of these two distances. One can unite the pursuers into a single player, which we call the first player. The second player is identified with the evader, he maximizes the cost function. The motivation for this problem is related [1, 4] to a space pursuit problem where the instants T_1 (T_2) are rendezvouz instants for the first (second) pursuer with the evader along their nominal trajectories. In [2, 3], the authors distinguished and numerically studied qualitatively different possibilities for solving this problem.

The simplest case is the case of "strong" pursuers when both pursuers exceed the evader in their dynamic capabilities. The hardest are cases when the dynamic advantage passes from the evader to the pursuers or vice versa during the game. Here, in particular, there appear level sets of the value function whose time sections are not single-connected, but as the reverse time increases they become single-connected again.

The essential problem here is to construct optimal (or quasioptimal) methods of control for the players with the feedback principle. The existing ideology of the theory of differential games presupposes for this case either storing the entire value function or its fast computation in the neighborhood of the current position. The optimal control in this case is constructed with some version of the generalized gradient for the value function [5–10].

In linear differential games with a convex payoff function, a simpler approach to constructing optimal control is possible [11–13], an approach that employs switching lines and surfaces. When we speak of switching lines or surfaces, we mean a partition of the phase space at every time instant into regions in each of which the controlling influence takes one of its limit values. Only boundaries of these regions are stored in memory, without the values of the value function. In a problem with two pursuers and one evader, the payoff function is not convex, but in [2, 3] the authors attempted to construct optimal (quasioptimal) strategies for the players anyway, with switching lines. For the case of strong pursuers, the corresponding statements related to the proof of optimality are shown in [2].

In this work, we consider the case of "weak" pursuers. We assume that $T_1 = T_2$. For this case, we formulate and prove statements regarding a quasioptimal control method for the first player based on switching lines. The method is stable with respect to numerical errors and measurement errors for the current phase state of the system. The case of weak pursuers for $T_1 \neq T_2$ is somewhat more complex, and we do not consider it here.

We note that currently there are many publications that study group pursuit problems [14–25]. These problems are hard primarily due to the large dimension of the phase vector and the fact that the payoff function is nonconvex. Therefore, these works usually make rather strong assumptions regarding the objects' dynamics (for instance, they consider objects with simple motions), their initial states and so on. In this work, where the number of objects is small, we attempt to get an exact solution without any significant simplifications.

2. PROBLEM SETTING

Pursuers P_1, P_2 and evader E move along a straight line. The dynamics of pursuers is given by the following relations:

$$\begin{aligned} &\ddot{z}_{P_1} = a_{P_1}, && |u_1| \leqslant \mu_1, && \ddot{z}_{P_2} = a_{P_2}, && |u_2| \leqslant \mu_2, \\ &\dot{a}_{P_1} = (u_1 - a_{P_1})/l_{P_1}, && a_{P_1}(t_0) = 0; && \dot{a}_{P_2} = (u_2 - a_{P_2})/l_{P_2}, && a_{P_2}(t_0) = 0. \end{aligned} \tag{1}$$

Here z_{P_1} and z_{P_2} are geometric coordinates of the pursuers; a_{P_1} and a_{P_2} are their accelerations caused by controls u_1 and u_2. Time constants l_{P_1} and l_{P_2} determine how fast the control is processed. The dynamics of the evader E is similar:

$$\ddot{z}_E = a_E, \quad \dot{a}_E = (v - a_E)/l_E, \quad |v| \leqslant \nu, \quad a_E(t_0) = 0. \tag{2}$$

To compare dynamic capabilities, we introduce parameters [1, 4] $\eta_i = \mu_i/\nu$, $\varepsilon = l_E/l_{P_i}$, $i = 1, 2$. In this work, we study the case of weak pursuers, when $\eta_i \leqslant 1$, $\eta_i\varepsilon_i \leqslant 1$, $i = 1, 2$, and for every i at least one of these inequalities is strict.

Let us fix an instant T. At this instant, we compute pursuer misses with respect to the evader:

$$r_{P_1,E}(T) = \left|z_E(T) - z_{P_1}(T)\right|, \quad r_{P_2,E}(T) = \left|z_E(T) - z_{P_2}(T)\right|. \tag{3}$$

Suppose that pursuers are coordinated. We unite them into a single player P whom we will call the *first player*. Player P has a vector control $u = (u_1, u_2)^{\mathrm{T}}$. Here and in what follows the superscript T denotes transposition. We regard the evader as the *second player*. As the payoff function we take the minimum of two misses:

$$\varphi = \min\{r_{P_1,E}(T),\ r_{P_2,E}(T)\}. \tag{4}$$

At every instant t, both players have exact information regarding all phase coordinates z_{P_1}, $\dot{z}_{P_1}$, a_{P_1}, z_{P_2}, $\dot{z}_{P_2}$, a_{P_2}, z_E, $\dot{z}_E$, a_E. We denote the vector composed of these values by z. The first player chooses his feedback control in order to minimize the value of the payoff function φ, while the second player maximizes the payoff.

We will assume that the game occurs during the interval $[\bar{t}, T]$, where $\bar{t} < T$. Let $Y = [\bar{t}, T] \times R^2$ be the game space.

Following [6, 8], as admissible first player strategies we consider arbitrary functions $(t, z) \mapsto U(t, z)$ with values in the set $\{(u_1, u_2) : |u_1| \leqslant \mu_1, |u_2| \leqslant \mu_2\}$. We denote by $z(\,\cdot\,; t_0, x_0, U, \Delta, v(\cdot))$ the step-by-step motion of system (1), (2) from position (t_0, x_0), when the first player applies strategy U in the discrete control scheme with step $\Delta > 0$, and the second player implements a measurable control $v(\cdot)$ with values $|v(t)| \leqslant \nu$. The term "discrete scheme" means the following. We choose a grid of moments t_s with a certain step Δ (starting from the instant t_0). Knowing at time instant t_s the system position $z(t_s)$, the first player computes his control $u= U\big(t_s, z(t_s)\big)$. This control remains constant until time instant $t_{s+1}= t_s + \Delta$. In position $\big(t_{s+1}, z(t_{s+1})\big)$, a new value for the controlling influence is computed, and so on.

Let

$$\Gamma(t_0, z_0, U, \Delta) = \sup_{v(\cdot)} \varphi\big(z(T; t_0, z_0, U, \Delta, v(\cdot))\big).$$

Here the supremum is taken over all measurable functions $t \mapsto v(t)$ bound by the constraint $|v(t)| \leqslant \nu$. The value $\varphi\big(z(T)\big)$ is the value of the payoff function (3), (4) at the termination instant T on the motion $z(\,\cdot\,; t_0, z_0, U, \Delta, v(\cdot))$.

The value $\Gamma(t_0, z_0, U, \Delta)$ has the meaning of the first player's guaranteed result by strategy U for the starting position (t_0, z_0) in a discrete control scheme with step Δ. The best guaranteed result for the first player for starting position (t_0, z_0) is given by formula

$$\Gamma(t_0, z_0) = \min_{U} \varlimsup_{\Delta \to 0} \Gamma(t_0, z_0, U, \Delta),$$

where $\varlimsup$ denotes the upper limit. It is shown in [6, 8] that the minimum over admissible strategies U is reached.

It is known that the best guaranteed result $\Gamma(t_0, z_0)$ for the first player coincides with the symmetrically defined best guaranteed result for the second player. Therefore, the value $\Gamma(t_0, z_0)$ is also called the value $\boldsymbol{V}(t_0, z_0)$ of the value function at point (t_0, z_0).

In this work we will show how to construct a quasioptimal (i.e., close to optimal in guaranteed result) first player strategy that would be suitable for all starting positions and stable to errors in its numerical specification and measurement errors for the current system phase position.

3. PASSING TO A TWO-DIMENSIONAL DIFFERENTIAL GAME

We introduce difference geometric coordinates $y_1 = z_E - z_{P_1}$, $y_2 = z_E - z_{P_2}$. We rewrite motion Eqs. (1), (2) and the payoff function (3), (4):

$$\begin{array}{ll} \ddot{y}_1 = a_E - a_{P_1}, & \ddot{y}_2 = a_E - a_{P_2}, \\ \dot{a}_{P_1} = (u_1 - a_{P_1})/l_{P_1}, & \dot{a}_{P_2} = (u_2 - a_{P_2})/l_{P_2}, \\ \dot{a}_E = (v - a_E)/l_E, & |u_2| \leqslant \mu_2, \\ |u_1| \leqslant \mu_1,\ |v| \leqslant \nu, & \varphi = \min\{|y_1(T)|,\ |y_2(T)|\}. \end{array} \tag{5}$$

Phase variables of system (5) are y_1, $\dot{y}_1$, a_{P_1}, y_2, $\dot{y}_2$, a_{P_2}, a_E; u_1 and u_2 are first player controls; v is the second player control. The payoff function φ depends on coordinates y_1 and y_2 at the instant T.

The standard approach to studying linear differential games with fixed termination instant and payoff function depending on some subset of objective components in the phase vector at the termination instant includes passing to new phase variables (see, e.g., [6, 8]). These variables are understood as values of objective components predicted for the termination instant under zero players' controls. They are often called *zero effort miss coordinates* [4, 26]. In our case, we pass to new phase coordinates x_1 and x_2, where $x_i(t)$ is the value of y_i predicted at the termination instant T, $i = 1, 2$.

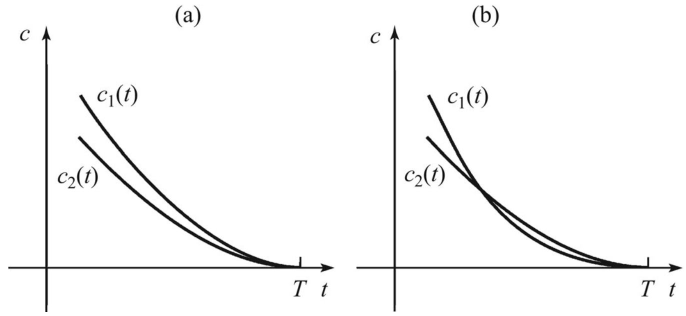

Fig. 1. Graphs of functions $c_1(t)$ and $c_2(t)$. Parameter values: (a) $\eta_1 < \eta_2 < 1$, $\eta_1\varepsilon_1 \leqslant \eta_2\varepsilon_2 \leqslant 1$; (b) $\eta_1 < \eta_2 < 1$, $\eta_2\varepsilon_2 < \eta_1\varepsilon_1 \leqslant 1$.

The formula for recomputing the coordinates is

$$x_i = y_i + \dot{y}_i\tau - a_{P_i} l_{P_i}^2 h(\tau/l_{P_i}) + a_E l_E^2 h(\tau/l_E), \quad i = 1, 2. \tag{6}$$

Here x_i, y_i, $\dot{y}_i$, a_{P_i}, and a_E depend on the time t; $\tau = T - t$. Function h is defined by relation $h(\alpha) = e^{-\alpha} + \alpha - 1$. We have $x_i(T) = y_i(T)$. We denote by $X(t, z)$ the two-dimensional vector composed of variables x_1 and x_2 defined according to (6).

The dynamics is written in new coordinates x_1 and x_2 as follows [1]:

$$\begin{aligned} \dot{x}_1 &= -l_{P_1} h(\tau/l_{P_1}) u_1 + l_E h(\tau/l_E) v, \qquad |u_1| \leqslant \mu_1, \\ \dot{x}_2 &= -l_{P_2} h(\tau/l_{P_2}) u_2 + l_E h(\tau/l_E) v, \qquad |u_2| \leqslant \mu_2, \ |v| \leqslant \nu. \end{aligned} \tag{7}$$

The payoff function has the form $\varphi(x_1(T), x_2(T)) = \min\{|x_1(T)|, \ |x_2(T)|\}$.

Let $x = (x_1, x_2)^{\mathrm{T}}$. Let $V(t, x)$ be the value function magnitude for game (7) in position (t, x). It is known that $\boldsymbol{V}(t, z) = V\big(t, X(t, z)\big)$. This relation allows us to find the value function for the original game (1)–(4) by using the value function of game (7). Transformation $(t, z) \mapsto x = X(t, z)$ also helps to recalculate feedback control in game (7) into the corresponding control in game (1)–(4).

For every $c \geqslant 0$ the level set $W_c = \big\{(t, x) : V(t, x) \leqslant c\big\}$ of the value function for game (7) is a solvability set in the considered game with result not exceeding c. This set also represents a maximal stable bridge [6, 8] in the game with dynamics (7) and terminal set $M_c = \big\{(T, x) : |x_1| \leqslant c, |x_2| \leqslant c\big\}$. We denote by $W_c(t) = \{x : (t, x) \in W_c\}$ the t-section of set W_c at time instant t.

When studying this problem, it is useful to consider individual games: $P_1 - E$ of the first pursuer against the evader and $P_2 - E$ of the second pursuer against the evader. These games are one-dimensional with respect to the phase variable. The dynamics of game $P_1 - E$ ($P_2 - E$) is given by the first (second) row in relations (7). The terminal payoff here is $|x_1(T)|$ ($|x_2(T)|$).

For individual games, we denote by $c_i(t)$, $i = 1, 2$, the value of the value function at instant t at point $x_i = 0$. The value $c_i(t)$ can be easily found by integrating the game dynamics $P_i - E$ on the interval $[t, T]$ for $u_i = +\mu_i$ and $v = +\nu$ with initial condition $x_i(t) = 0$ and computing the value $x_i(T)$. Different possibilities for the dependencies $t \mapsto c_1(t)$ and $t \mapsto c_2(t)$ are presented on Fig. 1. Apart from the common point for $t = T$, the graphs can have at most one more common point. In the case of identical pursuers we have $c_1(t) \equiv c_2(t)$.

We write system (7) in vector form:

$$\dot{x} = \mathcal{D}_1(t) u_1 + \mathcal{D}_2(t) u_2 + \mathcal{E}(t) v, \qquad |u_1| \leqslant \mu_1, \ |u_2| \leqslant \mu_2, \ |v| \leqslant \nu, \tag{8}$$

$$\mathcal{D}_1(t) = \big(-l_{P_1} h((T-t)/l_{P_1}), \, 0\big)^{\mathrm{T}}, \qquad \mathcal{D}_2(t) = \big(0, \, -l_{P_2} h((T-t)/l_{P_2})\big)^{\mathrm{T}},$$

$$\mathcal{E}(t) = \big(l_E h((T-t)/l_E), \, l_E h((T-t)/l_E)\big)^{\mathrm{T}}.$$

Further as the norm on the x_1, x_2 plane we take the norm $\|x\|_\infty = \max\{|x_1|, |x_2|\}$. The distance function is induced by this norm. We denote a closed α-neighborhood of a set A by $O(\alpha, A)$.

4. APPROXIMATING THE DIFFERENTIAL GAME

1. Together with system (8), we consider the approximating system

$$\dot{x} = D_1(t)u_1 + D_2(t)u_2 + E(t)v, \qquad |u_1| \leqslant \mu_1,\ |u_2| \leqslant \mu_2,\ |v| \leqslant \nu, \tag{9}$$

with piecewise constant functions

$$D_i(t) = \mathcal{D}_i(t_j), \quad E(t) = \mathcal{E}(t_j), \quad t \in [t_j, t_{j+1}), \quad i = 1, 2, \tag{10}$$

defined on some partition of the t axis with instants t_j. We use system (9) for numerical constructions. The payoff function for the approximating game is the same as for the game with dynamics (8). It satisfies the Lipshitz condition with constant $\lambda = 1$.

We denote by $x^{(1)}(t; t_*, x_*, u(\cdot), v(\cdot))$ (abbr. $x^{(1)}(t)$) the position of system (8) at the instant t if its motion at the instant t_* goes out of point x_* due to admissible measurable controls $u(\cdot)$ and $v(\cdot)$. Let $x^{(2)}(t; t_*, x_*, u(\cdot), v(\cdot))$ (abbr. $x^{(2)}(t)$) be a similar notation for the position of system (9). The discrepancy between motions $x^{(1)}(\cdot)$ and $x^{(2)}(\cdot)$ at an instant t, caused by the difference in dynamics (8) and (9), can be bounded from above by the value

$$\chi(t_*, t) = \sum_{i=1}^{2} \mu_i \int_{t_*}^{t} \|D_i(s) - \mathcal{D}_i(s)\|_\infty ds + \nu \int_{t_*}^{t} \|E(s) - \mathcal{E}(s)\|_\infty ds.$$

Let $V^{(2)}(t, x)$ be the value of the value function in the approximating game in position (t, x). Since the phase variable does not occur in the right-hand side of system (9), the Lipshitz constant for function $x \mapsto V^{(2)}(t, x)$ for any $t \leqslant T$ coincides [27, pp. 110–111] with the Lipshitz constant of the payoff function, i.e., coincides with the number $\lambda = 1$.

2. We also apply approximation (10) for individual one-dimensional differential games $P_1 - E$, $P_2 - E$. In order not to clutter the notation, we will use notation $c_i(t)$, $i = 1, 2$, introduced at the end of Section 3, also for the values $V^{(2)}_{P_i-E}(t, x_i)$ of the value function in approximating individual games at time moment t at point $x_i = 0$ (i.e., we write $c_i(t)$ instead of a more accurate notation $c_i^{(2)}(t)$). For approximating games, the relation between curves $t \mapsto c_i(t)$ remains the same as on Fig. 1. Let

$$\tilde{c}(t) = \min_{i=1,2} c_i(t), \qquad c'(t) = \max_{i=1,2} c_i(t). \tag{11}$$

Obviously, for every position (t, x) it holds that

$$V^{(2)}(t, x) \leqslant \min_{i=1,2} V^{(2)}_{P_i-E}(t, x_i).$$

Consequently, for every point x on the vertical (horizontal) axis we have $V^{(2)}(t, x) \leqslant c_1(t)$ $\big(V^{(2)}(t, x) \leqslant c_2(t)\big)$.

Suppose that for some instant t the minimum in (11) is achieved for $i = 1$. Let us show that for every x on the x_2 (x_1) axis it holds that

$$V^{(2)}(t, x) = \tilde{c}(t) \quad \big(V^{(2)}(t, x) \in [\tilde{c}(t), c'(t)]\big). \tag{12}$$

Indeed, for points on the x_2 (x_1) axis it holds that $V^{(2)}(t, x) \leqslant \tilde{c}(t)$ $\big(V^{(2)}(t, x) \leqslant c'(t)\big)$. Consider a point x on the positive part of the x_2 axis (x_1 axis). Let $v = +\nu$. Then on the interval $[t, T]$

motion of system (9) for any control $u(\cdot)$ will occur in the first quadrant, and at time instant T we get $x_2(T) \geqslant c_2(t)$ $(x_1(T) \geqslant c_1(t))$. Consequently,

$$V^{(2)}(t,x) \geqslant \min_{u(\cdot)} \min_{i=1,2} |x_i(T)| \geqslant \min_{i=1,2} c_i(t) = \tilde{c}(t).$$

Thus, for the considered point x on the x_2 (x_1) axis (12) holds. If point x lies on the negative part of the x_2 (x_1) axis, we let $v = -\nu$, and then the motion of system (9) occurs in the third quadrant. The considerations in this case are similar.

If at instant t the minimum in (11) is reached for $i = 2$ then $V^{(2)}(t,x) = \tilde{c}(t)$ for all points on the x_1 axis and $V^{(2)}(t,x) \in [\tilde{c}(t), c'(t)]$ for points on the x_2 axis.

In the case of identical pursuers (i.e., when $\mu_1 = \mu_2$, $l_{P_1} = l_{P_2}$) we have $c_1(t) \equiv c_2(t)$. In the case of different pursuers, without loss of generality we will assume that $\eta_1 < \eta_2$. If we also have $\eta_2\varepsilon_2 < \eta_1\varepsilon_1$ then there exists a unique instant $t^\nabla < T$ such that $c_1(t^\nabla) = c_2(t^\nabla)$. On the interval (t^∇, T) the minimum in (11) is achieved for $i = 1$; on the interval $(-\infty, t^\nabla)$, for $i = 2$. Correspondingly, for $t \in (t^\nabla, T)$ the value $V^{(2)}(t,x)$ on the x_2 axis equals $\tilde{c}(t)$, and for $t < t^\nabla$ the value $V^{(2)}(t,x)$ on the x_1 axis is $\tilde{c}(t)$. If $\eta_1\varepsilon_1 \leqslant \eta_2\varepsilon_2$ then for every $t < T$ the minimum in (11) is achieved for $i = 2$. The value $V^{(2)}(t,x)$ on the x_1 axis equals $\tilde{c}(t)$.

3. To find function $V^{(2)}$, we use a numerical algorithm for backward construction of t-sections $W_c^{(2)}(t) = \{x : V^{(2)}(t,x) \leqslant c\}$ for its level sets. The algorithm that takes into account the specifics of a plane has been developed by S.A. Ganebnyi. A description of the backward procedure is presented in [2, 3].

In this work, we give numerically constructed examples for a game with the following parameters:

$$\eta_1 = 0.7, \quad \eta_2 = 0.95, \quad \varepsilon_1 = 1.3, \quad \varepsilon_2 = 0.4, \quad T = T_1 = T_2 = 15. \tag{13}$$

Figure 2 shows the evolution of set $W_c^{(2)}(t)$ for $c = 5.0$ with time. We denote by the symbol $\tau = T - t$ the reverse time. The top left picture corresponds to the instant $\tau = 0$ when the game halts. The level set section at this instant represents a cross with infinite crossbeams. The top middle picture shows an intermediate reverse time instant, when infinite crossbeams have not yet disappeared but have become thinner. The top right picture shows the reverse time instant slightly later than the instant $\tau_1 = T - t_1$ when horizontal crossbeams disappear. At the instant τ_1, value of the value function on the horizontal axis equals 5.0, except for the points of the axis that are internal points for the set $W_{5.0}^{(2)}(t_1)$. On the left picture in the second row, we can see how angles of the set become more flat. The central picture shows the t-section configuration after the instant $\tau_2 = T - t_2$, when the vertical infinite band disappears, and the t-section breaks down into two trapezoids. At the instant τ_2, the value of the value function on the vertical axis is the same and equals 5.0. Then the trapezoids become pentagons (right picture in the second row). Slanted sides of the pentagons reduce, and finally the pentagons become rectangles (bottom left picture). Rectangles continue to reduce until the level set's t-section becomes empty.

Figure 3a shows the picture of $W_c^{(2)}(t)$ sections computed at the instant $t = 9.35$ $(\tau = 5.65)$ for the set of values c in the range from 0 to 40 with step $\Delta c = 1.0$. A similar picture of the sections for $t = 1.65$ $(\tau = 13.35)$ is shown on Fig. 3b. The first of the considered instant exceeds the instant $t^\nabla \approx 4.35$; the second one comes before.

4. For every $c \geqslant 0$ and $t \leqslant T$, the set $W_c^{(2)}(t)$ (if it is nonempty) is symmetric with respect to the origin of the x_1, x_2 plane, since both dynamics (9) (together with control constraints) and the payoff function possess this property. In the case of identical pursuers we additionally have a symmetry with respect to the bisecting line of the second and fourth quadrants.

We denote by $c_{\min}(t)$ the value of the minimum of function $V^{(2)}(t,\cdot)$ on the x_1, x_2 plane at time moment t. For $c \in [c_{\min}(t), \tilde{c}(t))$ the set $W_c^{(2)}(t)$ consists of two bounded disjoint subsets. Let

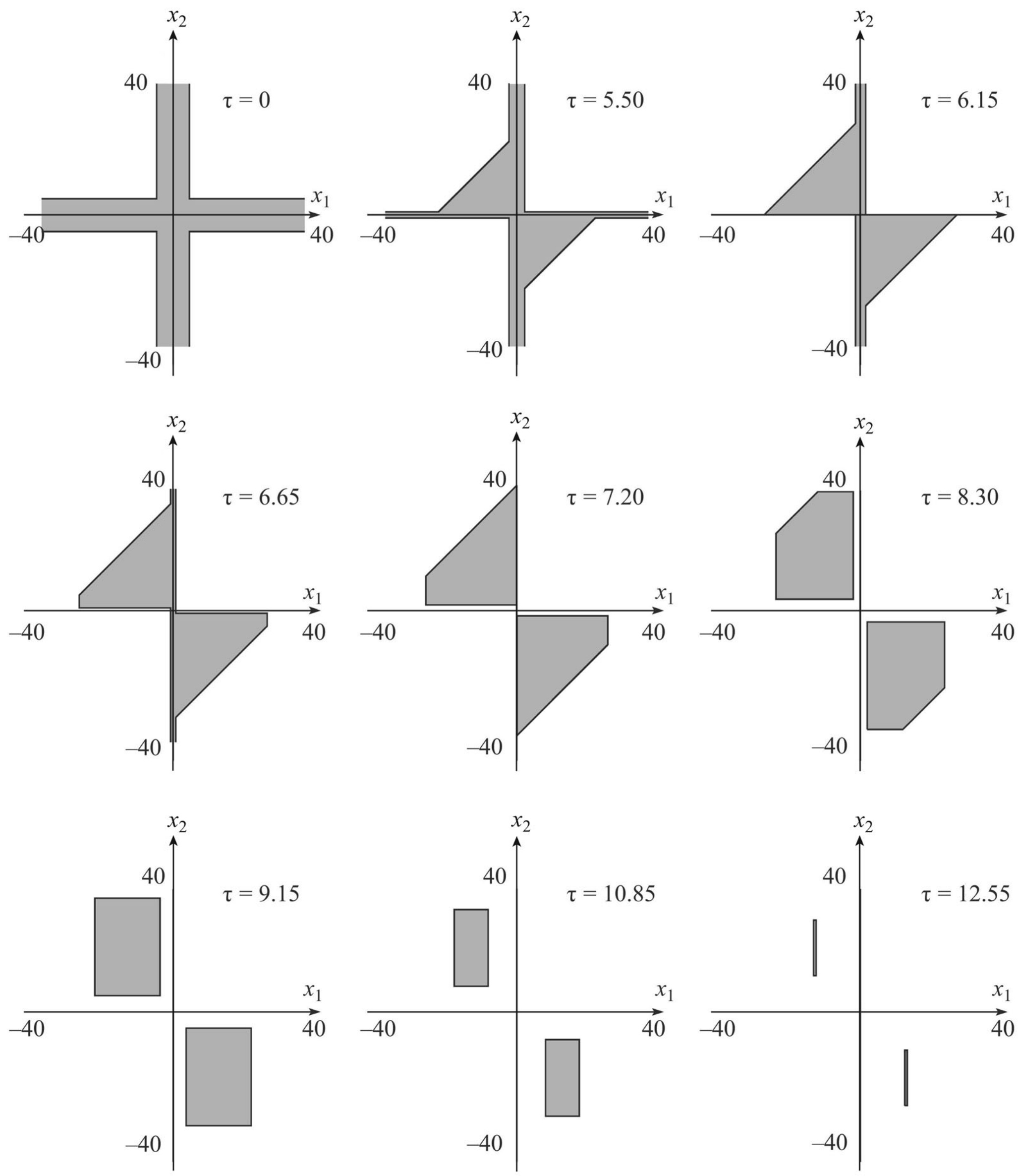

Fig. 2. Change of the set $W^{(2)}_{5.0}(t)$. Symbol $\tau = T - t$ denotes reverse time.

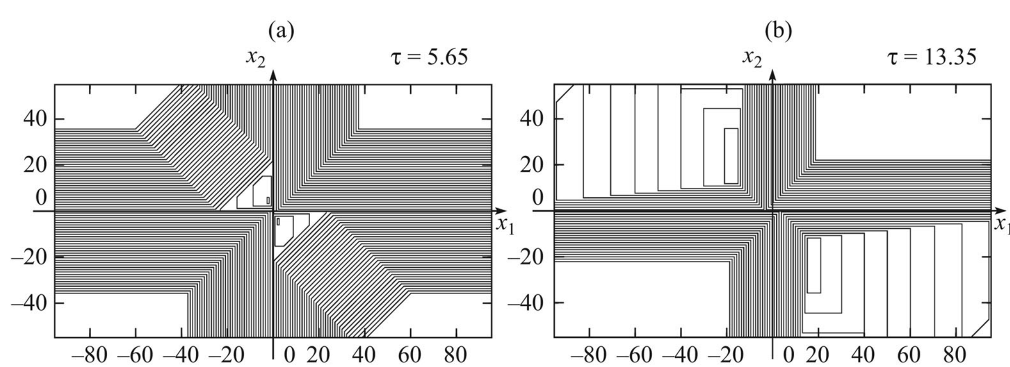

Fig. 3. t-sections of the level sets of the value function: (a) $\tau = 5.65$; (b) $\tau = 13.35$.

$W^{(2)}_{c,\mathrm{II}}(t)$ ($W^{(2)}_{c,\mathrm{IV}}(t)$) be the subset located at the second (fourth) quadrant. Let

$$Z(t) = W^{(2)}_{c_{\min}(t)}(t), \quad Z_{\mathrm{II}}(t) = W^{(2)}_{c_{\min}(t),\mathrm{II}}(t), \quad Z_{\mathrm{IV}}(t) = W^{(2)}_{c_{\min}(t),\mathrm{IV}}(t).$$

In the case $\eta_1 < \eta_2$, for $\eta_1\varepsilon_1 \leqslant \eta_2\varepsilon_2$ sets $Z_{\mathrm{II}}(t)$ and $Z_{\mathrm{IV}}(t)$ are vertical segments. If $\eta_2\varepsilon_2 < \eta_1\varepsilon_1$ then for small τ they are horizontal segments. As τ increases, these sets turn to points and then become vertical segments. As t decreases, the set $Z(t)$ always gets further from the origin, and the minimal value $c_{\min}(t)$ of the value function increases.

An important property of system (8) is that the directions of vectors $\mathcal{D}_1(t)$ and $\mathcal{D}_2(t)$ do not change with time. Vectors $D_1(t)$ and $D_2(t)$ that approximate system (9) also have this property, namely vector $D_1(t)$ ($D_2(t)$) is directed horizontally (vertically) opposite to the direction of the x_1 (x_2) axis. In particular, this property causes new horizontal and vertical plates to appear at the boundary of set $W^{(2)}_c(t)$ when it becomes discontinuous.

5. SWITCHING LINES

Suppose now that pursuers P_1 and P_2 are different, and $\eta_1 < \eta_2$. In the special case of identical pursuers our constructions will be simplified; see the corresponding remark at the end of the section.

1. Under the assumption that $\eta_2\varepsilon_2 < \eta_1\varepsilon_1$, let us consider an arbitrary instant $t \in (t^{\nabla}, T)$. Slicing the plane x_1, x_2 with horizontal lines, we see that the minimum of the restriction of function $V^{(2)}(t,\cdot)$ on each such line is realized either at a point or on a segment. In particular, if the straight line does not intersect the set $\mathrm{ci}\, W^{(2)}_{\tilde{c}(t)}(t) = \mathrm{cl}\left(\mathrm{int}\, W^{(2)}_{\tilde{c}(t)}(t)\right)$ (here int denotes the interior of the set, cl denotes closure) then the minimum is realized at a point on the x_2 axis.

If a straight line intersects the set $\mathrm{int}\, W^{(2)}_{\tilde{c}(t)}(t)$ but does not intersect the set $Z(t)$ of the global minimum of function $V^{(2)}(t,\cdot)$, then the minimum is reached on the segment which is for some $c \in (c_{\min}(t), \tilde{c}(t))$ a plate on the boundary of set $W^{(2)}_c(t)$. For a straight line going through the set $Z(t)$ the minimum of the restriction is reached on the intersection of the straight line with this set. In any case, as we get further from the minimum segment, the value function increases. At the edges of the horizontal line, the value function is constant.

These facts have been established by a careful study of the results of our numerical construction of the value function's level sets in game (9).

For $t \in (t^{\nabla}, T)$, we define the continuous switching line $\Pi(1,t)$ for control u_1 as follows. We begin by defining it in the second quadrant in the set $\mathrm{ci}\, W^{(2)}_{\tilde{c}(t)}(t)$. The set $Z_{\mathrm{II}}(t)$ is either a horizontal or a vertical segment. In the latter case we include it in the line $\Pi(1,t)$. At the boundary of the set $W^{(2)}_{c,\mathrm{II}}(t)$, $c \in [c_{\min}(t), \tilde{c}(t))$ we distinguish an upper horizontal plate. It belongs, either partially or completely, to the upper side of the rectangle $O(\alpha, Z_{\mathrm{II}}(t))$ for some $\alpha \geqslant 0$. We note the middle point of the intersection of such two segments and take it as a point on the line $\Pi(1,t)$. Below the $Z_{\mathrm{II}}(t)$ segment, we define the next portion of the switching line as a vertical line segment that ends at the boundary of the set $W^{(2)}_{\tilde{c}(t)}(t)$. Each of its points belongs to the lower horizontal plate on the boundary of some set $W^{(2)}_{c,\mathrm{II}}(t)$, $c \in [c_{\min}(t), \tilde{c}(t))$. Then we prolong the switching line with a horizontal segment that follows the boundary of set $W^{(2)}_{\tilde{c}(t)}(t)$ up until the vertical axis. In the fourth quadrant, we define the switching line $\Pi(1,t)$ in the set $\mathrm{ci}\, W^{(2)}_{\tilde{c}(t)}(t)$ similarly to the above, symmetric with respect to the origin. Outside the set $\mathrm{ci}\, W^{(2)}_{\tilde{c}(t)}(t)$ we assume that the line $\Pi(1,t)$ follows the x_2 axis.

Results of our numerical constructions for the line $\Pi(1,t)$ are presented on Fig. 4a. Constructions have been made on some grid of level sets $W^{(2)}_c$ for the function $V^{(2)}$. We show 10 sections $W^{(2)}_c(t)$.

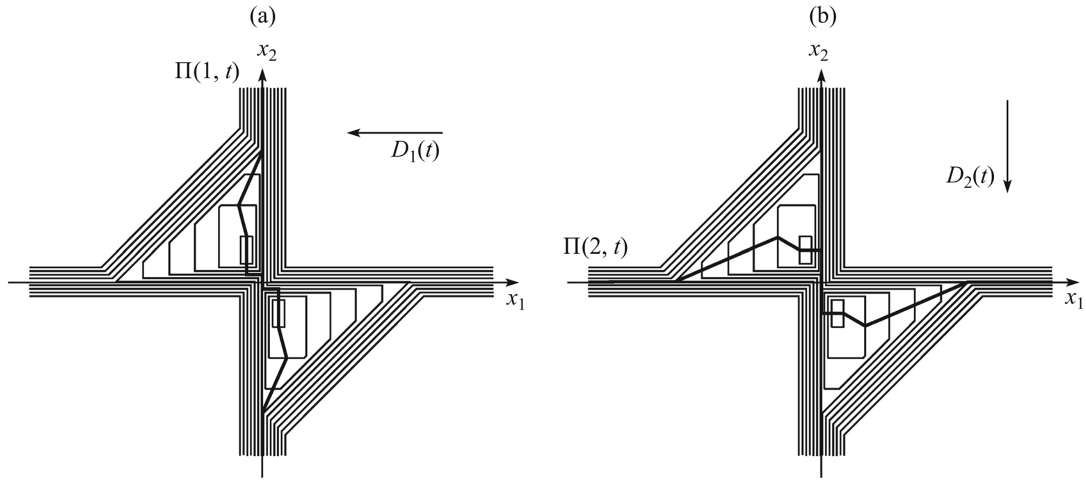

Fig. 4. Switching lines for an instant $t \in (t^{\nabla}, T)$: (a) $\Pi(1, t)$, (b) $\Pi(2, t)$.

The fourth section in increasing order corresponds to the value $c = \tilde{c}(t)$; the sixth, to the value $c = c'(t)$.

The behavior of the value function restriction to vertical lines is in many respects similar. The value function restriction is minimized either at a point or on a segment. One exception is the x_2 axis; on this axis, the value of the value function is constant and equals $\tilde{c}(t)$. For vertical straight lines that do not intersect the set $\operatorname{ci} W^{(2)}_{c'(t)}(t)$, the minimum is realized at the point where the line intersects with the x_1 axis. If the vertical line does intersect the set $\operatorname{int} W^{(2)}_{c'(t)}(t)$ but does not intersect the set $Z(t)$, the minimum is reached on a plate located at the boundary of set $W^{(2)}_c(t)$ for some $c \in (c_{\min}(t), c'(t))$. For a straight line passing through $Z(t)$ the minimum is realized at the intersection of the line with this set. As in the case of horizontal straight lines, as we go further away from the interval, the value function minimum increases. There also exist intervals where it is constant; two such infinite intervals lie at the ends of the line.

We define the switching line $\Pi(2, t)$ for $t \in (t^{\nabla}, T)$ in the second quadrant in the set $\operatorname{ci} W^{(2)}_{c'(t)}(t)$ to the left from segment $Z_{\mathrm{II}}(t)$, using middle points of intersections of the left plates at the boundaries of sets $W^{(2)}_c(t)$, $c \in [c_{\min}(t), c'(t))$, with left sides of the rectangles $O(\alpha, Z_{\mathrm{II}}(t))$, $\alpha \geqslant 0$. If a segment $Z_{\mathrm{II}}(t)$ is horizontal, we include it in the line $\Pi(2, t)$. To the right from the interval $Z_{\mathrm{II}}(t)$, we take a segment of the switching line as a horizontal segment that reaches up to the vertical axis. In the fourth quadrant, in the set $\operatorname{ci} W^{(2)}_{c'(t)}(t)$ we define the switching line $\Pi(2, t)$ similar to the above, symmetrically with respect to the origin. Outside the set $\operatorname{ci} W^{(2)}_{c'(t)}(t)$ we assume the line $\Pi(2, t)$ to go along the x_1 axis. Result of a numerical construction of the $\Pi(2, t)$ line is shown on Fig. 4b.

If $t \leqslant t^{\nabla}$ for $\eta_2\varepsilon_2 < \eta_1\varepsilon_1$ and if $t < T$ for $\eta_1\varepsilon_1 \geqslant \eta_2\varepsilon_2$, then switching lines $\Pi(1, t)$ and $\Pi(2, t)$ are defined with horizontal and vertical lines, taking into account that the value $\tilde{c}(t)$ is realized on the horizontal axis x_1, and the value $c'(t)$ corresponds to the value of the value function at the ends of the x_2 axis.

We denote by $\Pi_+(1, t)$ (respectively $\Pi_-(1, t)$) the part of the plane located strictly to the right (strictly to the left) of the switching line $\Pi(1, t)$. If $x \in \Pi_+(1, t)$ ($x \in \Pi_-(1, t)$) then control $u_1 = +\mu$ ($u_1 = -\mu$) directs the vector $D_1(t)u_1$ to the switching line, i.e., to the side where the value function $V^{(2)}(t, \cdot)$ decreases. We introduce a similar notation $\Pi_+(2, t)$ (respectively $\Pi_-(2, t)$) for the part of the plane above (below) the switching line $\Pi(2, t)$.

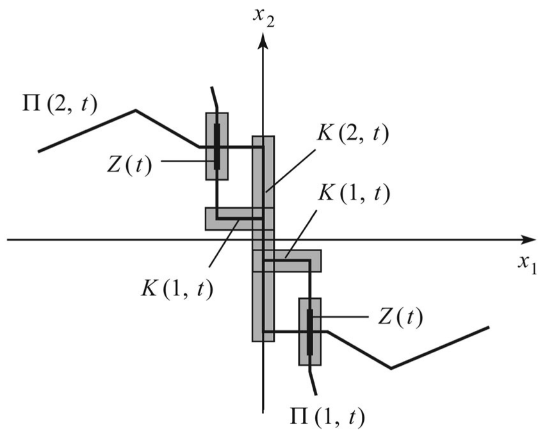

Fig. 5. Sets $Z(t)$, $K(1,t)$, $K(2,t)$ and their α-neighborhoods for an instant $t \in (t^\nabla, T)$. Segments $Z(t)$ of the global minimum of the value function $V^{(2)}(t,\cdot)$ are highlighted by line width.

2. In what follows, we treat lines $\Pi(1,t)$ and $\Pi(2,t)$ as found exactly for the approximating system (9). It will become clear that they define optimal feedback control in system (9) and quasioptimal (i.e., close to optimal) feedback control in system (8). In reality, we cannot construct perfect switching lines $\Pi(1,t)$ and $\Pi(2,t)$ numerically. For instance, even if we construct the sets $W_c^{(2)}(t)$ exactly, we still operate with some step with respect to parameter c. As a result, we get polylines that only approximate the perfect switching line, so there arises a problem of the guarantee that they provide for the first player.

3. For every $t < T$ and every horizontal (vertical) line passing through the point x, we denote by $\mathcal{V}(1,t,x)$ (respectively $\mathcal{V}(2,t,x)$) the minimal value of the value function $V^{(2)}(t,\cdot)$ on this line. We have $\mathcal{V}(1,t,x) = V^{(2)}(t,x)$ for $x \in \Pi(1,t)$ and $\mathcal{V}(2,t,x) = V^{(2)}(t,x)$ for $x \in \Pi(2,t)$.

We fix a number $r \geqslant 0$ and "extend" the line $\Pi(1,t)$, superimposing the centers of horizontal segments of length $2r$ on it. We denote the resulting set by $\Pi^r(1,t)$. Similarly, but with vertical segments, we define the set $\Pi^r(2,t)$.

We introduce this geometric r-extension for perfect switching lines in order to cover the case when they are constructed imprecisely. By this extension, we aim to "encircle" switching lines $\Pi(1,t)$ and $\Pi(2,t)$ with regions that "cover up" construction errors or imprecisions in the system position measurement. Here for control u_1 it is convenient to use the horizontal extension since on any horizontal line the value $\mathcal{V}(1,t,x)$ remains the same for all x on this line. If as a result of computations we get a value of $\mathcal{V}(1,t,x_*)$ at a certain point x_* at an instant t, and point x_* is horizontally apart from the line $\Pi(1,t)$ by at most distance r, we get an upper bound on the value $V^{(2)}(t,x_*)$: $V^{(2)}(t,x_*) \leqslant \mathcal{V}(1,t,x_*) + \lambda r$. Similarly, for control u_2 it is convenient to consider a vertical extension.

The above-described approach does not give an efficient "encircling" at an instant $t \in (t^\nabla, T)$ on two horizontal segments of the line $\Pi(1,t)$ and the (one) vertical segment of the line $\Pi(2,t)$. At time instant t^∇, two horizontal segments of the line $\Pi(1,t)$ converge into one segment located at the x_1 axis. For $t < t^\nabla$, the line $\Pi(1,t)$ has only one horizontal segment, while the line $\Pi(2,t)$ has two vertical segments.

We denote the above-mentioned segments by $K(1,t)$ for line $\Pi(1,t)$ and $K(2,t)$ for line $\Pi(2,t)$. It is very important that on the lines $K(1,t)$ and $K(2,t)$ the value of the value function remains the same and equals $\tilde{c}(t)$.

4. For what follows, we need to "prohibit" a fast transition of the motions of systems (8) and (9) from set $\Pi^r(1,t)$ to set $\Pi^r(2,t)$ and from set $\Pi^r(2,t)$ to set $\Pi^r(1,t)$ outside of some neighborhood of the intersection of lines $\Pi(1,t)$ and $\Pi(2,t)$. Let us consider the α-neighborhood $O(\alpha, Z(t))$ and $O(\alpha, K(i,t))$ of the sets $Z(t)$ and $K(i,t)$, $i = 1, 2$ (Fig. 5). Let

$$\Pi^r_\alpha(i,t) = \mathrm{cl}\left[\Pi^r(i,t) \backslash \left(O(\alpha, Z(t)) \bigcup O(\alpha, K(i,t))\right)\right], \quad \alpha \geqslant 0, \quad i = 1, 2.$$

Lines $\Pi(1,t)$ and $\Pi(2,t)$, $t < T$, depend on time continuously. Therefore, for every instant $\hat{t} \in [\bar{t}, T)$ there exist values $\hat{\alpha} > 0$ and $\hat{r} > 0$ such that

$$\Pi^r_\alpha(1,t) \bigcap \Pi^r_\alpha(2,t) = \varnothing, \quad t \in [\bar{t}, \hat{t}\,], \quad \alpha \geqslant \hat{\alpha}, \quad r \in [0, \hat{r}]. \tag{14}$$

Moreover, for the said values t, α, and r there exists a lower bound $\theta(\hat{t}, \hat{\alpha}, \hat{r}) > 0$ on the transition time of systems (8) and (9) from one of the sets $\Pi^r_\alpha(1, \cdot)$ and $\Pi^r_\alpha(2, \cdot)$ to another.

Remark. If pursuers are identical then $c_1(t) \equiv c_2(t)$. Consequently, $\tilde{c}(t) \equiv c'(t)$. All constructions become symmetrical with respect to the bisecting line of the second and fourth quadrants. The set $K(1,t)$ is a segment on the x_2 axis centered at the origin, while the set $K(2,t)$ is a segment on the x_1 axis. The set $Z(t)$ consists of two points on the bisecting line of the second and fourth quadrants.

6. AUXILIARY STATEMENTS

We formulate two lemmas; their proofs are given in the Appendix. Lemmas will be used to prove theorems on the guaranteed estimate when the first player in system (8) employs the control method based on switching lines constructed in system (9).

Let $\sigma_i = \max\{\|D_i(t)\|_\infty : t \in [\bar{t}, T]\}$, $i = 1, 2$; $\sigma = \max\{\sigma_1, \sigma_2\}$; $\mu = \max\{\mu_1, \mu_2\}$.

Lemma 1. *Fix $i = 1, 2$. Suppose that position $(t_*, x_*) \in Y$ and number $\delta > 0$, $t_* + \delta < T$, are such that $x_* \in \Pi_+(i, t_*)$ ($x_* \in \Pi_-(i, t_*)$), and any motion of system (9) starting at the instant t_* from point x_*, at every instant $t \in [t_*, t_* + \delta]$ remains in the set $\Pi_+(i,t)$ ($\Pi_-(i,t)$). Consider on the interval $[t_*, t_* + \delta]$ the motion $x^{(1)}(\cdot)$ of system (8) that at the instant t_* goes out of point x_* under the action of some second player control $v(\cdot)$ and first player control $u(\cdot)$ such that $u_i \equiv +\mu_i$ ($u_i \equiv -\mu_i$) except perhaps the interval $[t_*, t_* + \omega]$ of length $\omega \leqslant \delta$.*

Then for each $t \in [t_*, t_* + \delta]$ it holds that

$$\mathcal{V}(\bar{i}, t, x^{(1)}(t)) \leqslant V^{(2)}(t_*, x_*) + 2\lambda\omega\sigma_i\mu_i + \lambda\chi(t_*, t). \tag{15}$$

Here $\bar{i} = 2$ if $i = 1$, and $\bar{i} = 1$ if $i = 2$.

Comment. Suppose that $i = 2$, and we have chosen the $+$ sign among the variants $+$ and $-$. Then $x_* \in \Pi_+(2,t)$, and this agrees with our assumption regarding the form of the "correct" control $u_2(\cdot)$, which differs from $u_2 \equiv +\mu_2$ only on an interval of size ω. An admissible control $u_1(\cdot)$ is arbitrary. The value ω defines the value of the second term in the right-hand side of estimate (15). The third term is a standard addition that bounds from above the increment of the cost function $V^{(2)}$ caused by the difference in dynamics of systems (8) and (9).

Lemma 2. *Let $(t_*, x_*) \in Y$, $t^* \in (t_*, T)$ and $0 \leqslant \omega \leqslant t^* - t_*$. Suppose that along the motion $x^{(1)}(\cdot)$ of system (8) that at the instant t_* goes out of point x_* under the action of admissible controls $u(\cdot)$ and $v(\cdot)$ for each $i = 1, 2$ it holds that*

1) *either $x^{(1)}(t) \in \Pi_+(i,t)$ on the interval (t_*, t^*) and $u_i(t) = +\mu_i$ on $(t_* + \omega, t^*)$;*

2) *or $x^{(1)}(t) \in \Pi_-(i,t)$ on the interval (t_*, t^*) and $u_i(t) = -\mu_i$ on $(t_* + \omega, t^*)$.*

Then for each $t \in [t_*, t^*]$ it holds that

$$V^{(2)}(t, x^{(1)}(t)) \leqslant V^{(2)}(t_*, x_*) + 2\lambda\omega\sigma\mu + \lambda\chi(t_*, t). \tag{16}$$

7. GUARANTEED RESULT THEOREMS

7.1. Guaranteed Result Estimate for a Multivalued First Player Strategy

For $i = 1, 2$, $\alpha > 0$, $r \geqslant 0$, and $t \in [\bar{t}, T)$ we let

$$S(i,\alpha,r,t) = O(\alpha, Z(t)) \bigcup O(\alpha, K(i,t)) \bigcup \Pi^r_\alpha(i,t).$$

We introduce a multivalued strategy $(t,x) \mapsto \mathbf{U}(t,x)$ for the first player. We will assume that $\mathbf{U}_i(t,x) = \{u_i : |u_i| \leqslant \mu_i\}$ if $x \in S(i,\alpha,r,t)$, $i = 1,2$. Outside the set $S(i,\alpha,r,t)$, $t < T$, component $\mathbf{U}_i(t,x)$, $i = 1,2$, for strategy $\mathbf{U}$ is uniquely defined, namely: at position (t,x) the corresponding value u_i, $i = 1,2$, which is equal to $+\mu_i$ or $(-\mu_i)$, is chosen to be such that the vector $D_i(t)u_i$ is directed towards the set $S(i,\alpha,r,t)$. Note that for every point $x \notin S(i,\alpha,r,t)$ the horizontal direction from this point for $i = 1$ (vertical direction for $i = 2$) towards the set $S(i,\alpha,r,t)$ is unique and coincides with the direction towards the switching line $\Pi(1,t)$ $(\Pi(2,t))$.

Let us fix a time moment $\hat{t} \in [\bar{t}, T)$. By $\hat{t}$, we find $\hat{\alpha} > 0$ and $\hat{r} \in (0, \hat{\alpha})$ so that on $[\bar{t}, \hat{t}\,]$ a lower bound $\theta(\hat{t}, \hat{\alpha}, \hat{r}) > 0$ holds for the transition time for systems (8) and (9) from each of the sets $\Pi^{\hat{r}}_{\hat{\alpha}}(1,\cdot)$ and $\Pi^{\hat{r}}_{\hat{\alpha}}(2,\cdot)$ to another. Then bound $\theta(\hat{t}, \hat{\alpha}, \hat{r})$ on the transition time also hods for $\alpha \geqslant \hat{\alpha}$, $r \in [0, \hat{r}]$. In what follows we assume that these relations do hold. We fix an arbitrary positive $\vartheta < \theta(\hat{t}, \hat{\alpha}, \hat{r})$. We denote $\hat{c} = c'(\hat{t})$.

Suppose that the first player applies in system (8) the strategy $\mathbf{U}$ in the discrete scheme with step $\Delta \leqslant \vartheta$. At every instant t_s of the discrete scheme, the first player chooses control $u \in \mathbf{U}(t_s, x(t_s))$ and holds it constant on the interval $[t_s, t_s + \Delta)$.

Let us estimate the increment of function $V^{(2)}$ along the motion $x^{(1)}(\cdot)$ going at an instant $t_0 \in [\bar{t}, T)$ out from the point x_0 and generated by first player strategy $\mathbf{U}$ in the discrete scheme with step Δ together with some admissible second player control $v(\cdot)$.

Let $\Pi^r_\alpha(t) = \Pi^r_\alpha(1,t) \bigcup \Pi^r_\alpha(2,t)$ and $K(t) = K(1,t) \bigcup K(2,t)$.

1. We introduce the following time intervals.

1. Interval $\mathcal{T}_z = [t_z, t^z]$ from the time instant t_z when the point $x^{(1)}(t)$ first reaches the set $O(\alpha, Z(t))$ to the instant t^z when it last leaves it. If $\mathcal{T}_z = \varnothing$ we let $t^z = t_0$.

2. Interval $\mathcal{T}_k = [t_k, t^k]$ from the instant t_k when the point $x^{(1)}(t)$ first reaches the set $O(\alpha, K(t))$ to the instant t^k when it last leaves it. We consider this interval only when $t_k \in [t^z, \hat{t})$.

3. Interval $\mathcal{T}_{\hat{c}} = [t_{\hat{c}}, t^{\hat{c}}]$ from the instant $t_{\hat{c}}$ when the point $x^{(1)}(t)$ first reaches the set $O(\alpha, W^{(2)}_{\hat{c}}(t))$ for $t \geqslant \hat{t}$ to the instant $t^{\hat{c}}$ when it last leaves it.

4. Interval $\mathcal{T}_\flat = [t_\flat, t^\flat]$ for $t^\flat \leqslant \hat{t}$. We assume that $x^{(1)}(t_\flat) \in \Pi^r_\alpha(t_\flat)$ and $x^{(2)}(t^\flat) \in \Pi^r_\alpha(t^\flat)$. We also suppose that interval $\mathcal{T}_\flat$ is located to the right of the instant t^z and outside the interval $\mathcal{T}_k$. We assume that interval $\mathcal{T}_\flat$ has maximal length under all these assumptions.

2. We write estimates on the changes in function $V^{(2)}$ along the motion $x^{(1)}(\cdot)$. We denote by $\mathrm{Var}\,(V^{(2)}, [t_*, t^*])$ the increment of function $V^{(2)}$ on the interval $[t_*, t^*]$. We first consider intervals $\mathcal{T}_z$, $\mathcal{T}_k$, $\mathcal{T}_{\hat{c}}$.

At the instant t^z we have

$$V^{(2)}(t^z, x^{(1)}(t^z)) \leqslant c_{\min}(t^z) + \lambda\alpha \leqslant V^{(2)}(t_0, x_0) + \lambda\alpha. \tag{17}$$

For the instant t^k it holds that $V^{(2)}(t^k, x^{(1)}(t^k)) \leqslant \tilde{c}(t^k) + \lambda\alpha \leqslant \tilde{c}(t_k) + \lambda\alpha$. Since $\tilde{c}(t_k) \leqslant V^{(2)}(t_k, x^{(1)}(t_k)) + \lambda\alpha$, we get that

$$\mathrm{Var}\,(V^{(2)}, [t_k, t^k]) \leqslant 2\lambda\alpha. \tag{18}$$

For the instant $t^{\hat{c}}$ it holds that

$$V^{(2)}(t^{\hat{c}}, x^{(1)}(t^{\hat{c}})) \leqslant \hat{c} + \lambda\alpha. \tag{19}$$

3. It is not so easy to estimate the increment $V^{(2)}$ along the motion $x^{(1)}(\cdot)$ on the interval $\mathcal{T}_\flat$. To be definite, suppose that $x^{(1)}(t_\flat) \in \Pi^r_\alpha(1, t_\flat)$.

Let $t_1 = t_\flat$. We denote by t_{1+} the largest instant that belongs to time interval $[t_1, t_1 + \vartheta] \bigcap [t_1, t^\flat]$ for which $x^{(1)}(t) \in \Pi^r_\alpha(t)$. Since in time ϑ it is impossible to transit from set $\Pi^r_\alpha(1, \cdot)$ to set $\Pi^r_\alpha(2, \cdot)$, it means that $x^{(1)}(t_{1+}) \in \Pi^r_\alpha(1, t_{1+})$. To estimate the value $V^{(2)}(t_{1+}, x^{(1)}(t_{1+}))$ we use Lemma 1.

Suppose that $t_{1+} < t^\flat$. Let t_2 be the smallest time instant from the interval $[t_1 + \vartheta, t^\flat]$ such that $x^{(1)}(t) \in \Pi^r_\alpha(t)$. Here both the case when $x^{(1)}(t_2) \in \Pi^r_\alpha(1, t_2)$ and the case when $x^{(1)}(t_2) \in \Pi^r_\alpha(2, t_2)$ are possible. In both cases, the point $x^{(1)}(t)$ on time interval (t_{1+}, t_2) is located outside the set $S(1, \alpha, r, t) \bigcup S(2, \alpha, r, t)$. To estimate the value $\operatorname{Var}\left(V^{(2)}, [t_{1+}, t_2]\right)$, we use Lemma 2.

If $t_2 < t^\flat$, we introduce the time instant t_{2+} defined as the largest instant from the interval $[t_2, t_2 + \vartheta] \bigcap [t_2, t^\flat]$ such that $x^{(1)}(t) \in \Pi^r_\alpha(t)$. In case $x^{(1)}(t_2) \in \Pi^r_\alpha(1, t_2)$ we have $x^{(1)}(t_{2+}) \in \Pi^r_\alpha(1, t_{2+})$. In case $x^{(1)}(t_2) \in \Pi^r_\alpha(2, t_2)$ we have $x^{(1)}(t_{2+}) \in \Pi^r_\alpha(2, t_{2+})$. Suppose that $t_{2+} < t^\flat$. Then we introduce the time instant t_3 as the smallest instant from the interval $[t_2 + \vartheta, t^\flat]$ such that $x^{(1)}(t) \in \Pi^r_\alpha(t)$, and so on.

On intervals of the form $[t_j, t_{j+}]$, by Lemma 1 we get that

$$\mathcal{V}(i, t_{j+}, x^{(1)}(t_{j+})) \leqslant V^{(2)}(t_j, x^{(1)}(t_j)) + 2\lambda\Delta\sigma\mu + \lambda\chi(t_j, t_{j+}). \tag{20}$$

Here $i = 1$ if $x^{(1)}(t_j) \in \Pi^r_\alpha(1, t_j)$. Control u_2 under strategy $\mathbf{U}$ can be chosen "incorrectly" only on some interval $[t_j, t_j + \omega]$, where $\omega \leqslant \Delta$. If $x^{(1)}(t_j) \in \Pi^r_\alpha(2, t_j)$ then in the left-hand side of inequality (20) we take $i = 2$.

Passing from the value $\mathcal{V}(i, t_{j+}, x^{(1)}(t_{j+}))$ to the value $V^{(2)}(t_{j+}, x^{(1)}(t_{j+}))$, we have the following inequality: $V^{(2)}(t_{j+}, x^{(1)}(t_{j+})) \leqslant \mathcal{V}(i, t_{j+}, x^{(1)}(t_{j+})) + \lambda r$. Therefore,

$$\operatorname{Var}\left(V^{(2)}, [t_j, t_{j+}]\right) \leqslant 2\lambda\Delta\sigma\mu + \lambda r + \lambda\chi(t_j, t_{j+}). \tag{21}$$

For intervals of the form $[t_{j+}, t_{j+1}]$, based on Lemma 2 for $\omega \leqslant \Delta$ we get

$$\operatorname{Var}\left(V^{(2)}, [t_{j+}, t_{j+1}]\right) \leqslant 2\lambda\Delta\sigma\mu + \lambda\chi(t_{j+}, t_{j+1}). \tag{22}$$

Due to relations (21) and (22) it holds that

$$\operatorname{Var}\left(V^{(2)}, [t_j, t_{j+1}]\right) \leqslant 4\lambda\Delta\sigma\mu + \lambda r + \lambda\chi(t_j, t_{j+1}).$$

On the interval $[t_\flat, t^\flat]$ we have at most $[(t^\flat - t_\flat)/\vartheta]$ (here and in what follows $[\cdot]$ denotes the whole part of a number) of complete intervals of the form $[t_j, t_{j+1}]$. The last interval that ends at time instant $t^\flat$ can be an interval of the form $[t_j, t_{j+}]$, where $t_{j+} - t_j \leqslant \vartheta$. We get the bound

$$\operatorname{Var}\left(V^{(2)}, [t_\flat, t^\flat]\right) \leqslant \left(\left[\frac{t^\flat - t_\flat}{\vartheta}\right] + 1\right)(4\lambda\Delta\sigma\mu + \lambda r) + \lambda\chi(t_\flat, t^\flat). \tag{23}$$

4. Interval $[t^z, \hat{t}\,]$ can contain at most two intervals of the form $\mathcal{T}_\flat$ divided by the interval $\mathcal{T}_k$. We denote the first of them by $[t_{\flat 1}, t^{\flat 1}]$; the second, by $[t_{\flat 2}, t^{\flat 2}]$. On intervals $(t^z, t_{\flat 1})$, $(t^{\flat 1}, t_k)$, $(t^k, t_{\flat 2})$, and $(t^{\flat 2}, \hat{t})$ the point $x^{(1)}(t)$ is located outside the set $S(1, \alpha, r, t) \bigcup S(2, \alpha, r, t)$. Therefore, on each such interval we can estimate the increment of function $V^{(2)}$ with Lemma 2, letting $\omega \leqslant \Delta$. By doing so and taking into account estimates (17), (18), and (23), we get that

$$\operatorname{Var}(V^{(2)}, [t_0, \hat{t}]) \leqslant \left(\left[\frac{\hat{t} - t_0}{\vartheta} \right] + 2 \right) (4\lambda\Delta\sigma\mu + \lambda r) + 4 \times 2\lambda\Delta\sigma\mu + 3\lambda\alpha + \lambda\chi(t_0, \hat{t}). \tag{24}$$

5. Consider the case when on the interval $[\hat{t}, T)$ the point $x^{(1)}(t)$ falls into the set $O(\alpha, W^{(2)}_{\hat{c}}(t))$ (in particular, $x^{(1)}(\hat{t}) \in O(\alpha, W^{(2)}_{\hat{c}}(\hat{t}))$ if $\hat{t} \in \mathcal{T}_k$). At the instant $t^{\hat{c}}$ we have estimate (19). For $t \geqslant t^{\hat{c}}$ point $x^{(1)}(t)$ is located outside the set $O(\alpha, W^{(2)}_{\hat{c}}(t))$. Since $Z(t) \subset W^{(2)}_{\hat{c}}(t)$, $K(t) \subset W^{(2)}_{\hat{c}}(t)$ and $r \leqslant \alpha$, for $t \geqslant t^{\hat{c}}$ the motion $x^{(1)}(\cdot)$ proceeds outside the set $S(1, \alpha, r, t) \bigcup S(2, \alpha, r, t)$. Therefore, this motion is subject to the "correct" first player control with respect to components u_i, $i = 1, 2$, except maybe only some interval $[t^{\hat{c}}, t^{\hat{c}} + \omega]$, where $\omega \leqslant \Delta$. Using Lemma 2, we get for $t \in [t^{\hat{c}}, T]$ an estimate

$$V^{(2)}(t, x^{(1)}(t)) \leqslant \hat{c} + \lambda\alpha + 2\lambda\Delta\sigma\mu + \lambda\chi(t^{\hat{c}}, t). \tag{25}$$

Suppose that on the interval $[\hat{t}, T)$ point $x^{(1)}(t)$ does not fall into the set $O(\alpha, W^{(2)}_{\hat{c}}(t))$. Then the motion $x^{(1)}(\cdot)$ proceeds for $t \geqslant \hat{t}$ outside the above-mentioned sets, and in estimate (24) we increase the last term.

6. Thus, the final estimate for the instant T has the form

$$V^{(2)}(T, x^{(1)}(T)) \leqslant \max\{F(T), L(T)\}, \tag{26}$$

$$F(T) = V^{(2)}(t_0, x_0) + \left(\left[\frac{\hat{t} - t_0}{\vartheta} \right] + 2 \right) (4\lambda\Delta\sigma\mu + \lambda r) + 8\lambda\Delta\sigma\mu + 3\lambda\alpha + \lambda\chi(t_0, T),$$

$$L(T) = \hat{c} + \lambda\alpha + 2\lambda\Delta\sigma\mu + \lambda\chi(t_0, T).$$

Since $V^{(2)}(T, x^{(1)}(T)) = \varphi(x^{(1)}_1(T), x^{(1)}_2(T))$, estimate (26) holds for the first player guaranteed result when he uses in system (8) the strategy $\mathbf{U}$ in the discrete control scheme with step Δ.

Theorem 1. *Consider an instant $\hat{t} \in [\bar{t}, T)$ and parameters $\hat{\alpha} > 0$ and $\hat{r} > 0$ such that there holds a lower bound $\theta(\hat{t}, \hat{\alpha}, \hat{r}) > 0$ on the transition time from each of the sets $\Pi^r_{\hat{\alpha}}(1, \cdot)$ and $\Pi^r_{\hat{\alpha}}(2, \cdot)$ to another. Let $\alpha \geqslant \hat{\alpha}$ and $r \in [0, \hat{r}]$. Suppose that the multivalued strategy $\mathbf{U}$ defined with respect to t on time interval $[\bar{t}, T)$ takes value $\mathbf{U}_i(t, x) = \{u_i : |u_i| \leqslant \mu_i\}$ in the set $S(i, \alpha, r, t)$, $i = 1, 2$. Suppose that outside the set $S(i, \alpha, r, t)$ the value of $\mathbf{U}_i(t, x)$, equal to $+\mu_i$ or $(-\mu_i)$, is chosen to be such that the vector $D_i(t)\mathbf{U}_i(t, x)$ is directed towards the set $S(i, \hat{\alpha}, r, t)$, $i = 1, 2$. Fix positive $\vartheta < \theta(\hat{t}, \hat{\alpha}, \hat{r})$. Then for every starting position $(t_0, x_0) \in Y$ the strategy $\mathbf{U}$ in the discrete control scheme with step $\Delta \leqslant \vartheta$ in system (8) guarantees the first player a result given by formula (26).*

7.2. Stability of the Proposed Control Method

For $i = 1, 2$, $t \in [\bar{t}, \hat{t}]$, and $\beta > 0$, in the neighborhood $O(\beta, \Pi(i, t))$ we draw an arbitrary continuous line $\pi(i, t)$ which we will use to construct the component U^*_i of the first player strategy U^*. Let x be some state at the instant t. Consider a ray with direction $\mathcal{D}_i(t)$ outgoing from this point. If it intersects the line $\pi(i, t)$, we let $U^*_i(t, x) = +\mu_i$; otherwise we take $U^*_i(t, x) = -\mu_i$.

Take arbitrary $\varepsilon > 0$. We choose the instant $\hat{t}$ from condition $\hat{c} = c'(\hat{t}) = \varepsilon/4$. Suppose that the number $\hat{\alpha}$ satisfies relation $3\lambda\hat{\alpha} = \varepsilon/2$. We fix a number $\hat{r} \in (0, \hat{\alpha}]$ so that on interval $[\bar{t}, \hat{t}]$ there exists a lower bound $\theta(\hat{t}, \hat{\alpha}, \hat{r}) > 0$ on the transition time for motions of systems (8) and (9) from set $\Pi^{\hat{r}}_{\hat{\alpha}}(1,\cdot)$ to set $\Pi^{\hat{r}}_{\hat{\alpha}}(2,\cdot)$ and vice versa, from $\Pi^{\hat{r}}_{\hat{\alpha}}(2,\cdot)$ to $\Pi^{\hat{r}}_{\hat{\alpha}}(1,\cdot)$. Fix arbitrary positive $\vartheta < \theta(\hat{t},\hat{\alpha},\hat{r})$. We find $r^* \in (0,\hat{r}]$ and $\Delta^* \leqslant \vartheta$ such that

$$\left(\left[\frac{T-\bar{t}}{\vartheta}\right]+2\right)(4\lambda\Delta^*\sigma\mu+\lambda r^*)+8\lambda\Delta^*\sigma\mu \leqslant \frac{\varepsilon}{2}.$$

Let $\xi_1(\hat{t})$ $(\xi_2(\hat{t}))$ be a lower bound on the angle under which the line $\Pi(1,t)\backslash K(1,t)$ $(\Pi(2,t)\backslash K(2,t))$ intersects horizontal (vertical) straight lines at any time instant $t\in[\bar{t},\hat{t}\,]$. Let $\xi(\hat{t}) = \min\{\xi_1(\hat{t}),\xi_2(\hat{t})\}$.

Let $\beta^* = r^*\sin\xi(\hat{t})/\sqrt{2}$. One can show that $\beta \leqslant \beta^*$ satisfies the inclusions $O(\beta,\Pi(i,t)) \subset S(i,\hat{\alpha},r^*,t)$, $i=1,2$, $t\in[\bar{t},\hat{t}\,]$. The strategy U^* corresponding to a value β is a uniquely defined sample of multivalued strategy $\mathbf{U}$ defined by $\hat{\alpha}$ and r^*.

Due to (26) and the definition of values $\hat{t}$, $\hat{\alpha}$, r^*, Δ^*, and β^*, we get that for $\beta\in[0,\beta^*]$ strategy U^* in the discrete scheme with step $\Delta\in(0,\Delta^*]$ guarantees for every starting position $(t_0,x_0)\in Y$ an estimate

$$\varphi(x_1^{(1)}(T),x_2^{(1)}(T)) \leqslant V^{(2)}(t_0,x_0)+\varepsilon+\lambda\chi(t_0,T). \tag{27}$$

Estimates (26) and (27) are done for the case when to construct his control the first player at an instant t knows the exact position $x^{(1)}(t)$ of system (8). Consider the case of imprecise measurements. Suppose that first player instead of the true value $x^{(1)}(t)$ gets a measurement $\zeta(t)$ such that $\left\|\zeta(t)-x^{(1)}(t)\right\|_\infty \leqslant h$. The player uses this measurement to compute his control $U^*(t,\zeta(t))$. The following statement holds.

Theorem 2. *For every $\varepsilon>0$ one can find numbers $\gamma^*>0$, $h^*>0$, and $\Delta^*>0$ such that if the strategy U^* in system* (8) *is constructed based on switching lines $\pi(1,t)$ and $\pi(2,t)$ located for every $t\in[\bar{t},T)$ inside the sets $O(\gamma^*,\Pi(1,t))$ and $O(\gamma^*,\Pi(2,t))$, the measurement error does not exceed h^* and the step $\Delta>0$ of the discrete control scheme satisfies inequality $\Delta\leqslant\Delta^*$, then for every starting position $(t_0,x_0)\in Y$ and every realization $v(\cdot)$ of the second player control estimate* (27) *holds.*

To prove the statement, take $\gamma^*\leqslant\beta^*/2$ and $h^*\leqslant\beta^*/2$.

8. MODELING RESULTS

To show the modeling results, we consider the motion of pursuers P_1, P_2 and evader E on a two-dimensional plane. We call this plane the *original geometric space*. Suppose that in this motion, the horizontal component of the velocity vector for each object remains constant. Suppose that these components are such that the instants of horizontal passage of objects P_1, E and objects P_2, E are the same and equal to T. Thus, controlling influences only work on the horizontal shift. The dynamics of side motion is given by relations (1) and (2); the resulting miss is given by formula (4). Figure 6 denotes the horizontal axis by d. The d coordinate shows the longitudinal position of objects.

Game parameters are chosen according to (13). Starting side velocities and accelerations are assumed to be zero: $\dot{z}^0_{P_1}=\dot{z}^0_{P_2}=\dot{z}^0_E=0$, $a^0_{P_1}=a^0_{P_2}=a^0_E=0$. The initial time instant $t_0=0$.

The first player controls with the help of switching lines constructed in the approximating system (9). He uses exact knowledge of all phase coordinates for the pursuers and the evader.

Figures 6a presents the trajectories of objects for initial values of side deviations $z_{P_1}(t_0)=50$, $z_{P_2}(t_0)=-30$. The second player control is realized with his switching lines, also constructed in system (9). The construction procedure for the second player's switching lines is described in [2]. Whether the second player control method based on his switching lines is close to optimal is a problem that has not yet been carefully analyzed.

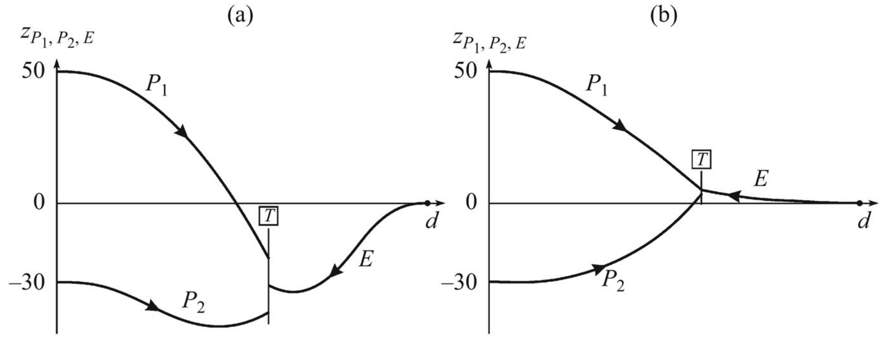

Fig. 6. Object trajectories in the original geometric space: (a) when both players use quasioptimal controls; (b) when the first player uses quasioptimal control and the second player uses random control.

Figure 6b shows the situation for the same initial deviations but for a random second player control (on each step of the discrete control scheme, the second player uses a random control from the interval $[-\nu, +\nu]$). Here the second player is caught exactly.

9. CONCLUSION

Control based on switching lines in a game problem that reduces to a two-dimensional (with respect to the phase variable) differential game with fixed termination instant presupposes a partition of the phase plane at every time instant into "cells" in whose internal points the controlling influence is constant and takes one of the limit values. It is important what information related to the value function should be given on switching lines that define the partition. In this work, for a differential game with two pursuers and one evader we find a set of parameters for the problem (the case of "weak" pursuers) when no additional information, apart from the switching lines themselves, is required to implement a suboptimal control method for the minimizing player. The proposed approach to constructing a quasioptimal strategy is stable with respect to small computational errors in the construction of switching lines and also to informational errors in determining the current phase state of the system. Previously a similar result has been obtained for the case of "strong" pursuers, but the case of weak pursuers is much harder.

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APPENDIX

Proof of Lemma 1. To be definite, suppose that $i = 2$ and we have chosen the $+$ sign from the $+$ and $-$ signs.

Together with the motion $x^{(1)}(\,\cdot\,; t_*, x_*, u(\cdot), v(\cdot))$ of system (8), which in the formulation of Lemma 1 was denoted as $x^{(1)}(\cdot)$, we additionally consider the motion $x^{(2)}(\,\cdot\,; t_*, x_*, u(\cdot), v(\cdot))$ (abbreviated $x^{(2)}(\cdot)$) of system (9) that develops under the same controls $u(\cdot)$ and $v(\cdot)$. Let $c_* = V^{(2)}(t_*, x_*)$. Fix an arbitrary instant $t \in [t_*, t_* + \delta]$.

With an open-loop control $v(\cdot)$ considered on time interval $[t_*, t]$ we find an open-loop control $u_{\rm st}(\cdot)$ so that

$$x^{(2)}_{\rm st}(t) \in W^{(2)}_{c_*}(t), \tag{A.1}$$

where $x^{(2)}_{\rm st}(\cdot) = x^{(2)}_{\rm st}(\,\cdot\,; t_*, x_*, u_{\rm st}(\cdot), v(\cdot))$ is a motion of system (9) that at the instant t_* goes out of point x_* under the action of controls $u_{\rm st}(\cdot)$ and $v(\cdot)$. This can always be done with the stability property [6, 8] of the level set $W^{(2)}_{c_*}$ of the cost function $V^{(2)}$. Inclusion (A.1) means that

$$V^{(2)}(t, x^{(2)}_{\rm st}(t)) \leqslant V^{(2)}(t_*, x_*). \tag{A.2}$$

Consider a new control $\hat{u}_{\rm st}(\cdot)$ with components $\hat{u}_{1\,\rm st}(\cdot) = u_{1\,\rm st}(\cdot)$ and $\hat{u}_{2\,\rm st} \equiv +\mu_2$. Let $\hat{x}^{(2)}_{\rm st}(\cdot)$ be a motion of system (9) that at the instant t_* goes out of point x_* under the action of controls $\hat{u}_{\rm st}(\cdot)$ and $v(\cdot)$. The following relations hold:

$$\hat{x}^{(2)}_{1\,\rm st}(t) = x^{(2)}_{1\,\rm st}(t), \qquad \hat{x}^{(2)}_{2\,\rm st}(t) \leqslant x^{(2)}_{2\,\rm st}(t). \tag{A.3}$$

Since points $\hat{x}^{(2)}_{\rm st}(t)$ and $x^{(2)}_{\rm st}(t)$ are located in the set $\Pi_+(2, t)$, it means that (A.3) implies that

$$V^{(2)}(t, \hat{x}^{(2)}_{\rm st}(t)) \leqslant V^{(2)}(t, x^{(2)}_{\rm st}(t)). \tag{A.4}$$

By the conditions of Lemma 1, the $u_2(\cdot)$ component of vector control $u(\cdot)$ differs from a constant control $\hat{u}_{2\,\rm st}(t) \equiv +\mu_2$ only on an interval of length ω. Therefore

$$\left| x^{(2)}_2(t) - \hat{x}^{(2)}_{2\,\rm st}(t) \right| \leqslant 2\omega\sigma_2\mu_2. \tag{A.5}$$

We have

$$\left| x^{(1)}_2(t) - x^{(2)}_2(t) \right| \leqslant \chi(t_*, t). \tag{A.6}$$

Now (A.5) and (A.6) imply that

$$\left| x^{(1)}_2(t) - \hat{x}^{(2)}_{2\,\rm st}(t) \right| \leqslant 2\omega\sigma_2\mu_2 + \chi(t_*, t). \tag{A.7}$$

Consider a horizontal line passing through the point $x^{(1)}(t)$. Due to (A.7), this line contains a point a such that $\|a - \hat{x}^{(2)}_{\rm st}(t)\|_\infty \leqslant 2\omega\sigma_2\mu_2 + \chi(t_*, t)$. Consequently, $V^{(2)}(t, a) \leqslant V^{(2)}(t, \hat{x}^{(2)}_{\rm st}(t)) + 2\lambda\omega\sigma_2\mu_2 + \lambda\chi(t_*, t)$. This, together with (A.2) and (A.4), implies that $V^{(2)}(t, a) \leqslant V^{(2)}(t_*, x_*) + 2\lambda\omega\sigma_2\mu_2 + \lambda\chi(t_*, t)$.

Since $\mathcal{V}(1, t, a) \leqslant V^{(2)}(t, a)$,

$$\mathcal{V}(1, t, a) \leqslant V^{(2)}(t_*, x_*) + 2\lambda\omega\sigma_2\mu_2 + \lambda\chi(t_*, t). \tag{A.8}$$

Due to equality $\mathcal{V}(1, t, a) = \mathcal{V}(1, t, x^{(1)}(t))$, from (A.8) we get (15).

Before proving Lemma 2 we turn to Lemma 3.

Lemma 3. *Let $(t_*, x_*) \in Y$, $\delta > 0$, $t_* + \delta < T$ and $0 \leqslant \omega \leqslant t_* + \delta$. Suppose that any motion of system (9) starting at the instant t_* from point x_* for $t \in [t_*, t_* + \delta]$ does not reach the lines $\Pi(i, t)$, $i = 1, 2$.*

Suppose that along some motion $x^{(1)}(\cdot)$ of system (8), at the instant t_* outgoing from point x_* due to admissible controls $u(\cdot)$ and $v(\cdot)$, for every $i = 1, 2$ it holds that

1) *either $x^{(1)}(t) \in \Pi_+(i,t)$ on the interval $[t_*, t_* + \delta]$ and $u_i(t) = +\mu_i$ on $[t_* + \omega, t_* + \delta]$;*

2) *or $x^{(1)}(t) \in \Pi_-(i,t)$ on the interval $[t_*, t_* + \delta]$ and $u_i(t) = -\mu_i$ on $[t_* + \omega, t_* + \delta]$.*

Then for each $t \in [t_*, t_* + \delta]$ it holds that

$$V^{(2)}(t, x^{(1)}(t)) \leqslant V^{(2)}(t_*, x_*) + 2\lambda\omega\sigma\mu + \lambda\chi(t_*, t). \tag{A.9}$$

Proof of Lemma 3. From the possible combinations of signs $+$, $-$ and values $i = 1, 2$ we consider the case when along motion $x^{(1)}(\cdot)$ of system (8) on the interval $[t_*, t_* + \delta]$ it holds that $x^{(1)}(t) \in \Pi_+(i,t)$, $i = 1, 2$. Let $c_* = V^{(2)}(t_*, x_*)$. Fix an arbitrary instant $t \in [t_*, t_* + \delta]$.

For an open-loop control $v(\cdot)$ considered on the interval $[t_*, t]$, we find an open-loop control $u_{\mathrm{st}}(\cdot)$ so that

$$x^{(2)}_{\mathrm{st}}(t) \in W^{(2)}_{c_*}(t), \tag{A.10}$$

where $x^{(2)}_{\mathrm{st}}(\cdot) = x^{(2)}_{\mathrm{st}}(\,\cdot\,; t_*, x_*, u_{\mathrm{st}}(\cdot), v(\cdot))$ is a motion of system (9) that at the instant t_* goes out from point x_* under the action of controls $u_{\mathrm{st}}(\cdot)$ and $v(\cdot)$. Inclusion (A.10) means that $V^{(2)}(t, x^{(2)}_{\mathrm{st}}(t)) \leqslant V^{(2)}(t_*, x_*)$.

We introduce a new control $\hat{u}_{\mathrm{st}}(\cdot)$ with components $\hat{u}_{i\,\mathrm{st}}(t) = +\mu_i$, $i = 1, 2$. In our case, it holds that $\hat{x}^{(2)}_{1\,\mathrm{st}}(t) \leqslant x^{(2)}_{1\,\mathrm{st}}(t)$ and $\hat{x}^{(2)}_{2\,\mathrm{st}}(t) \leqslant x^{(2)}_{2\,\mathrm{st}}(t)$. Since points $\hat{x}^{(2)}_{\mathrm{st}}(t)$ and $x^{(2)}_{\mathrm{st}}(t)$ belong to $\Pi_+(1,t) \bigcap \Pi_+(2,t)$, we have that

$$V^{(2)}(t, \hat{x}^{(2)}_{\mathrm{st}}(t)) \leqslant V^{(2)}(t, x^{(2)}_{\mathrm{st}}(t)) \leqslant V^{(2)}(t_*, x_*). \tag{A.11}$$

By the conditions of Lemma 3, the $u_i(\cdot)$ component differs from $u_i \equiv +\mu_i$ only on an interval of length at most ω. Therefore, $\left|x^{(2)}_i(t) - \hat{x}^{(2)}_{i\,\mathrm{st}}(t)\right| \leqslant 2\omega\sigma_i\mu_i$, $i = 1, 2$. Thus,

$$\left\|x^{(2)}(t) - \hat{x}^{(2)}_{\mathrm{st}}(t)\right\|_\infty \leqslant 2\omega\sigma\mu.$$

We have $\left\|x^{(1)}(t) - x^{(2)}(t)\right\|_\infty \leqslant \chi(t_*, t)$. Consequently,

$$\left\|x^{(1)}(t) - \hat{x}^{(2)}_{\mathrm{st}}(t)\right\|_\infty \leqslant 2\omega\sigma\mu + \chi(t_*, t).$$

Due to (A.11) this implies (A.9).

Proof of Lemma 2. Fix $t \in (t_*, t^*)$. Let $t_\diamond \in (t_*, t)$. We find a sufficiently fine-grained partition of segment $[t_\diamond, t]$ by instants $\{t_j\}$, $j = 1, 2, \ldots, e$, $t_1 = t_\diamond$, $t_e = t$, $t_{j+1} \leqslant t_j + \delta$, such that for every interval $[t_j, t_{j+1}]$, $j = 1, 2, \ldots, e-1$, an arbitrary motion of system (9) that at the instants t_j goes out of point $x^{(1)}(t_j)$ does not reach a switching line on this interval. This can be achieved since switching lines change continuously in time, and also due to the assumption we made regarding the location of motion $x^{(1)}(\cdot)$ with respect to switching lines.

Let $\rho(t_j) = 0$ if $t_j \leqslant t_* + \omega$, and $\rho(t_j) = 1$ if $t_j < t_* + \omega$. Due to Lemma 3, for each j we have the following relation:

$$V^{(2)}(t_{j+1}, x^{(1)}(t_{j+1})) \leqslant V^{(2)}(t_j, x^{(1)}(t_j)) + \rho(t_j)2\lambda\delta\sigma\mu + \lambda\chi(t_j, t_{j+1}). \tag{A.12}$$

Applying estimate (A.12) for $j = 1, 2, \ldots, e-1$, we obtain the inequality

$$V^{(2)}(t, x^{(1)}(t)) \leqslant V^{(2)}(t_\diamond, x^{(1)}(t_\diamond)) + 2\lambda(\omega + \delta)\sigma\mu + \lambda\chi(t_\diamond, t).$$

Passing to the limit for $\delta \to 0$, and then for $t_\diamond \to t_*$, we get estimate (16) for $t \in (t_*, t^*)$. If $t = t^*$ we add one more passage to the limit.

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