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CONTROL
IN DETERMINISTIC SYSTEMS

Three-Dimensional Reachability Set for a Nonlinear Control System

V. S. Patsko, S. G. Pyatko, and A. A. Fedotov

Institute of Mathematics and Mechanics, Ural Division, Russian Academy of Sciences, ul. S. Kavalevskoi 16, Yekaterinburg, 620219 Russia

Academy of Civil Aviation, St. Petersburg, Russia

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Abstract—A third-order nonlinear control system governing automobile or aircraft motion in a horizontal plane is considered. A theorem on the number and character of switchings of the controls that lead to the boundary of the reachability set is proved. Examples of numerical construction of the reachability set are presented.

INTRODUCTION

In the literature on mathematical control theory, there are few examples of third-order nonlinear systems for which the reachability sets in the three-dimensional space of phase coordinates have been constructed. This is explained by the fact that the reachability sets for nonlinear systems, as a rule, are not convex, which considerably complicates the analytical description and numerical construction of these sets.

A reachability set $G(T)$ at a fixed time T is a set of all states in the phase space that can be reached at the time T from a given initial state by means of an admissible control.

In this paper, results of studying the reachability set for a third-order nonlinear controlled system are presented. In this system, two coordinates describe the geometric position in the plane and the third coordinate is the angle specifying the direction of the velocity vector. The magnitude of the velocity is assumed to be constant. A scalar control determines the instantaneous angular rate of rotation of the linear velocity vector, and the absolute value of the rate is assumed to be bounded. Such a system is often used as the simplest model of automobile or aircraft motion in a horizontal plane [1–5].

By applying the Pontryagin maximum principle [6] to this system, it is not difficult to show that each point belonging to the boundary of the set $G(T)$ can be reached by means of a piecewise constant control with a finite number of switchings. In this paper, a theorem on the number and character of the control switchings is proved. This theorem is used for constructing numerically the boundary of the reachability set.

The set $G(T)$ being examined provides a nontrivial example of a reachability set in the three-dimensional space for a nonlinear control system. The set $G(T)$ constructed in this work can be used as a test in the development of universal numerical algorithms for constructing reachability sets for nonlinear control systems. The results obtained can also be useful when analyzing procedures for constructing prediction sets in problems with incomplete information for systems modeling aircraft motion in a horizontal plane [7, 8].

1. PROBLEM STATEMENT

Let the plane motion of a controlled object be governed by the following system of differential equations:

$$\dot{x} = V\cos\varphi,$$
$$\dot{y} = V\sin\varphi,$$
$$\dot{\varphi} = \frac{k}{V}u, \quad |u| \le 1, \qquad (1.1)$$
$$V = \text{const} > 0, \quad k = \text{const} > 0.$$

Here, x and y are coordinates of the geometric position, φ is the angle of the velocity vector with the abscissa axis (Fig. 1), V is the magnitude of the velocity, and k is the maximal lateral acceleration. Admissible controls $u(\cdot)$ are measurable functions of time satisfying the constraint $|u| \le 1$. The angle φ runs over the values in the interval $(-\infty, \infty)$.

The phase vector (x, y, φ) of system (1.1) is denoted by z. For the sake of brevity, we introduce the notation $\alpha = k/V$.

Let z_0 be an arbitrary fixed state of system (1.1) at an initial instant t_0. The reachability set $G(T)$ for $T \ge t_0$ is a set of all points z of the three-dimensional phase space that can be reached at the instant T from the initial point z_0 by means of an admissible control defined on the interval $[t_0, T]$.

Since the coefficients of system (1.1) do not depend on time, the selection of the initial instant t_0 does not matter. In addition, the initial state z_0 can also be taken arbitrarily: the reachability sets for different initial states are obtained from one another by a rotation and translation. It follows from the general results of mathematical control theory [9] that the set $G(T)$ is closed and bounded.

The aim of this work is to prove a statement on the number and character of the switchings of the controls that bring the system to the boundary of the set $G(T)$ and to numerically construct this boundary.

It should be noted that the projection of the set $G(T)$ onto the plane x, y is considered in the paper [10]. The authors are not aware of works where a three-dimensional reachability set for system (1.1) is considered.

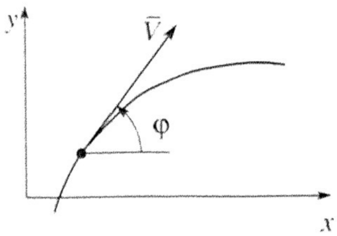

Fig. 1. The coordinate system.

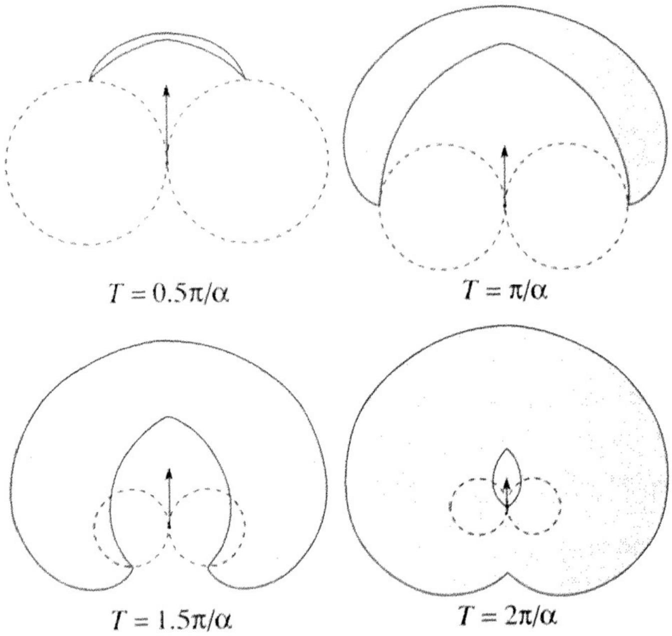

Fig. 2. Projections of the reachability set onto the plane x, y.

2. PROJECTION OF THE REACHABILITY SET ONTO THE PLANE OF GEOMETRIC COORDINATES

Before we proceed to examining the reachability set in the three-dimensional phase space, we show its projection onto the plane x, y. Projections of the reachability set for $T_i = i0.5\pi/\alpha$, $i = 1, 2, 3, 4$, are depicted in Fig. 2 (assuming a zero initial instant). At the selected instants of time, the velocity vector is turned through the angle of $i0.5\pi$ ($-i0.5\pi$) if the control is $u = 1$ ($u = -1$). Note that the scales in all figures are generally different.

Trajectories corresponding to limiting controls $u = 1$ and $u = -1$ are circles of radius V/α. The initial velocity vector is marked by an arrow. The sets depicted in the figure are computed by means of relations given in [10].

3. MAXIMUM PRINCIPLE

It is known [9] that the controls that bring the system to the boundary of the reachability set satisfy the Pontryagin maximum principle. Let us write down the equations of the maximum principle for system (1.1).

Let $u^*(\cdot)$ be an admissible control and ($x^*(\cdot)$, $y^*(\cdot)$, $\varphi^*(\cdot)$) be the corresponding motion of system (1.1) on the interval $[t_0, t_*]$. The differential equations governing the adjoint system have the form

$$\begin{aligned} \dot{\psi}_1 &= 0, \\ \dot{\psi}_2 &= 0, \\ \dot{\psi}_3 &= \psi_1 V \sin\varphi^*(t) - \psi_2 V \cos\varphi^*(t). \end{aligned} \tag{3.1}$$

The maximum principle implies that there exists a nonzero solution ($\psi_1^*(\cdot)$, $\psi_2^*(\cdot)$, $\psi_3^*(\cdot)$) of system (3.1) that satisfies the condition

$$\begin{aligned} &\psi_1^*(t)V\cos\varphi^*(t) + \psi_2^*(t)V\sin\varphi^*(t) \\ &+ \psi_3^*(t)\alpha u^*(t) = \max_{|u| \le 1}[\psi_1^*(t)V\cos\varphi^*(t) \\ &+ \psi_2^*(t)V\sin\varphi^*(t) + \psi_3^*(t)\alpha u] \end{aligned}$$

almost everywhere in the interval $[t_0, t_*]$. Hence, the maximum condition is as follows:

$$\begin{aligned} &\text{almost everywhere } \psi_3^*(t)u^*(t) \\ &= \max_{|u| \le 1} \psi_3^*(t)u, \quad t \in [t_0, t_*]. \end{aligned} \tag{3.2}$$

Since the functions $\psi_1^*(\cdot)$ and $\psi_2^*(\cdot)$ are constant, we will use the notation ψ_1^* and ψ_2^* for them. If $\psi_1^* = 0$ and $\psi_2^* = 0$, then $\psi_3^*(t) = \text{const} \neq 0$ on the interval $[t_0, t_*]$. Hence, in this case, either $u^*(t) = 1$ almost everywhere or $u^*(t) = -1$ almost everywhere.

Now, let at least one of the functions, ψ_1^* or ψ_2^*, be nonzero. It follows from (1.1) and (3.1) that $\psi_3^*(t)$ can be written as

$$\psi_3^*(t) = \psi_1^* y^*(t) - \psi_2^* x^*(t) + C.$$

Hence, it follows that $\psi_3^*(t) = 0$ if and only if the point ($x^*(t)$, $y^*(t)$) of the geometric position at any time t satisfies the straight line equation

$$\psi_1^* y - \psi_2^* x + C = 0. \tag{3.3}$$

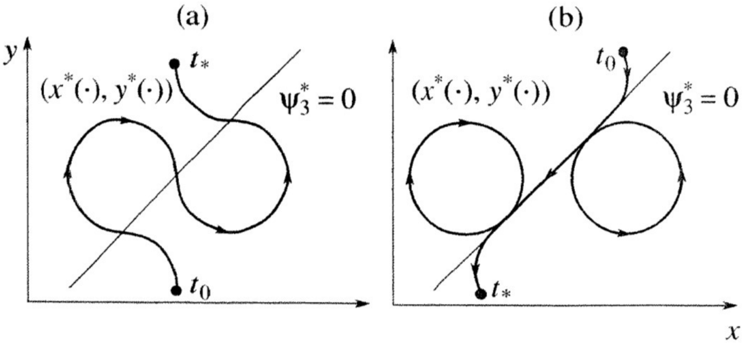

Fig. 3. Trajectories satisfying the maximum principle and the switching line.

Equation (3.3) was considered in many works (e.g., [3, 11]) that employed the maximum principle for studying system (1.1).

By virtue of (3.2), if $\psi_3^*(t) > 0$ ($\psi_3^*(t) < 0$), then $u^*(t) = 1$ ($u^*(t) = -1$) almost everywhere in this interval. In this case, the projection of the phase point onto the plane x, y moves counterclockwise (clockwise) along an arc of a circle of radius V/α in the half-plane $\psi_1^* y - \psi_2^* x + C > 0$ ($\psi_1^* y - \psi_2^* x + C < 0$). A fragment of the motion of duration $2\pi/\alpha$ with the control satisfying the condition $u^*(t) = 1$ or $u^*(t) = -1$ almost everywhere in the corresponding time interval will be referred to as a cycle. The projection of the trajectory of a cycle onto the plane x, y is a circle. If $\psi_3^*(t) = 0$ on a certain interval of time, then the trajectory of the corresponding motion $(x^*(\cdot), y^*(\cdot))$ on this interval is the straight line (3.3). Hence, $\varphi^*(t) = \text{const}$, and $u^*(t) = 0$ almost everywhere in this interval.

Consider the possible variants of the relative position of the trajectory of motion $(x^*(\cdot), y^*(\cdot))$ and the straight line (3.3).

1. The trajectory intersects the straight line (3.3) at a certain instant at a nonzero angle (Fig. 3a). In this case, the trajectory can be represented as a set of circle arcs such that the time intervals between two adjacent instants of the intersection with the straight line (3.3) are equal to each other. The function $\psi_3^*(\cdot)$ changes sign on the interval $[t_0, t_*]$ a finite number of times.

2. The straight line (3.3) is tangent to the trajectory at a certain instant (Fig. 3b). In this case, the trajectory can be represented as a set of circle arcs and linear segments. The linear segments belong to the straight line (3.3), and the circle arcs are tangent to this line. Note that any inner complete circle segment is one or several cycles following each other. The sign of the function $\psi_3^*(\cdot)$ on the interval $[t_0, t_*]$ either does not change or changes a finite number of times.

3. The trajectory does not intersect the straight line (3.3). In this case, the sign of the function $\psi_3^*(\cdot)$ does not change on the entire interval $[t_0, t_*]$, and the trajectory is an arc of a circle.

Thus, if the maximum condition (3.2) is satisfied, the function $\psi_3^*(\cdot)$ may change sign on the interval $[t_0, t_*]$ only a finite number of times. Therefore, for the control $u^*(\cdot)$ that generates the motion $z^*(\cdot)$ and satisfies the maximum principle, we can take a piecewise constant control taking values 0, ± 1 with a finite number of switchings in the interval $[t_0, t_*]$. For definiteness, the control is assumed to be piecewise constant from the right. The instant t_* is not included in the number of the switching instants. The discussions in this section can be summarized as follows.

Lemma 1. Let a motion $z^*(\cdot)$ be generated by a piecewise constant control $u^*(\cdot)$ that satisfies the maximum principle. Then,

(a) if the motion $z^*(\cdot)$ does not contain fragments with zero control or cycles, the time intervals between two adjacent switching instants are identical;

(b) if the motion $z^*(\cdot)$ does not contain fragments with zero control but does contain at least one cycle, the geometric coordinates at the switching instants coincide;

(c) if the motion $z^*(\cdot)$ contains identical points of the geometric position at switching instants, it contains at least one cycle;

(d) if the motion $z^*(\cdot)$ contains a fragment with zero control, any complete inner fragment with the control equal to 1 or -1 is one or several cycles following each other.

4. PROPERTIES OF MOTIONS WITH PIECEWISE CONSTANT CONTROL

In this section, we consider piecewise constant controls taking values 0 or ± 1. As shown in Section 3, these controls are sufficient for constructing the boundary of the reachability set. In Lemmas 2 and 3, we analyze motions that do not contain fragments with zero control. In Lemma 4, the case where there is a fragment with zero control is considered. In what follows, the symbol ∂ denotes the boundary of a set and the symbol int denotes its interior.

Lemma 2. Let a motion $z(\cdot)$ on an interval $[t_0, t_*]$ be generated by a piecewise control $u(\cdot)$ taking values ± 1 with two switching instants t_1 and t_2. Suppose that the points of the geometric position in the plane x, y at the switching instants are different. In addition, let the inequality

$$(t_1 - t_0) + (t_* - t_2) > (t_2 - t_1) \tag{4.1}$$

be fulfilled. Then, $z(t_*) \in \operatorname{int} G(t_*)$.

Proof. Without loss of generality, we may assume that the sequence of values of the control $u(\cdot)$ is as follows: $-1, 1, -1$. Denote by (x_1, y_1) and (x_2, y_2) the points of the geometric position corresponding to the switching instants t_1 and t_2.

Assume the contrary; i.e., let $z(t_*) \in \partial G(t_*)$. Then, the control $u(\cdot)$ satisfies the maximum principle. Since the points (x_1, y_1) and (x_2, y_2) do not coincide, it follows from assertion (a) of Lemma 1 that the motion $z(\cdot)$ does not contain cycles on the interval $[t_0, t_*]$.

Let us select instants $\bar{t} \in (t_0, t_1)$ and $\hat{t} \in (t_2, t_*)$ that satisfy the equation

$$(t_1 - \bar{t}) + (\hat{t} - t_2) = (t_2 - t_1), \tag{4.2}$$

which is always possible on the strength of inequality (4.1).

Introduce the notation $\tilde{t}_1 = \bar{t} + \hat{t} - t_2$ and $\tilde{t}_2 = \hat{t} - t_1 + \bar{t}$. Let us consider an auxiliary motion $\tilde{z}(\cdot) = (\tilde{x}(\cdot), \tilde{y}(\cdot), \tilde{\varphi}(\cdot))$ defined on the interval $[t_0, t_*]$ with the initial condition $z(t_0)$ that is generated by the control

$$\tilde{u}(t) = \begin{cases} -1, & t \in [t_0, \bar{t}) \\ 1, & t \in [\bar{t}, \tilde{t}_1) \\ -1, & t \in [\tilde{t}_1, \tilde{t}_2) \\ 1, & t \in [\tilde{t}_2, \hat{t}) \\ -1, & t \in [\hat{t}, t_*]. \end{cases}$$

The trajectories of the original and auxiliary motions are schematically shown in Fig. 4.

On the half-interval $[\bar{t}, \hat{t})$, the control $u(\cdot)$ has three intervals of constancy, $[\bar{t}, t_1)$, $[t_1, t_2)$, and $[t_2, \hat{t})$, where it takes values -1, 1, and -1, respectively. The auxiliary control $\tilde{u}(\cdot)$ on the half-interval $[\bar{t}, \hat{t})$ also has three intervals of constancy, $[\bar{t}, \tilde{t}_1)$, $[\tilde{t}_1, \tilde{t}_2)$, and $[\tilde{t}_2, \hat{t})$, with the values 1, -1, and 1.

It is evident that

$$t_1 - \bar{t} = \hat{t} - \tilde{t}_2, \quad t_2 - t_1 = \tilde{t}_2 - \tilde{t}_1, \quad \hat{t} - t_2 = \tilde{t}_1 - \bar{t}.$$

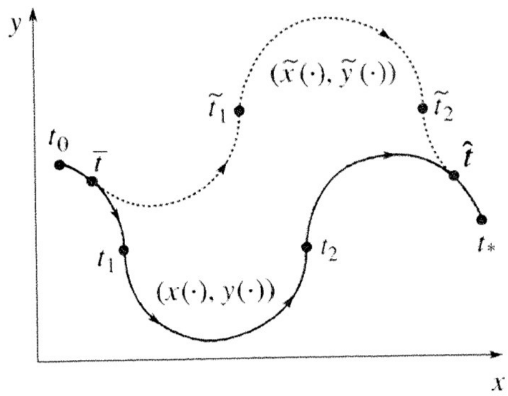

Fig. 4. Explanation of the proof of Lemma 2.

Note that, on the corresponding pairs of intervals, the controls of the original and auxiliary motions are opposite in sign. Hence, it follows that

$$u(t) = -\tilde{u}(\bar{t} + \hat{t} - t), \quad t \in (\bar{t}, t_1) \cup (t_1, t_2) \cup (t_2, \hat{t}). \tag{4.3}$$

Thus, the equality $u(t) = -\tilde{u}(\bar{t} + \hat{t} - t)$ is satisfied everywhere in the interval $(\bar{t}, \hat{t})$ except for the instants t_1 and t_2.

Using the third equation in system (1.1), the definition of the instants $\tilde{t}_1$ and $\tilde{t}_2$, and Eq. (4.2), we obtain

$$\tilde{\varphi}(\hat{t}) = \tilde{\varphi}(\bar{t}) + \alpha(\tilde{t}_1 - \bar{t}) - \alpha(\tilde{t}_2 - \tilde{t}_1) + \alpha(\hat{t} - \tilde{t}_2)$$
$$= \tilde{\varphi}(\bar{t}) + \alpha(2\tilde{t}_1 - \bar{t} - 2\tilde{t}_2 + \hat{t}) = \tilde{\varphi}(\bar{t}).$$

Hence, taking into account the equality $\tilde{\varphi}(\bar{t}) = \varphi(\bar{t})$, we obtain

$$\tilde{\varphi}(\hat{t}) = \varphi(\bar{t}). \tag{4.4}$$

Using (4.3), (4.4), and the third equation in (1.1), we obtain

$$\varphi(t) = \varphi(\bar{t}) + \int_{\bar{t}}^{t} \alpha u(\tau) d\tau = \tilde{\varphi}(\hat{t}) - \int_{\bar{t}}^{t} \alpha \tilde{u}(\bar{t} - \hat{t} - \tau) d\tau.$$

Changing the integration variable, $s = \bar{t} + \hat{t} - \tau$, we obtain

$$\varphi(t) = \tilde{\varphi}(\hat{t}) + \int_{\hat{t}}^{\bar{t} + \hat{t} - t} \alpha \tilde{u}(s) ds = \tilde{\varphi}(\bar{t} + \hat{t} - t).$$

Hence,

$$\varphi(t) = \tilde{\varphi}(\bar{t} + \hat{t} - t), \quad t \in [\bar{t}, \hat{t}]. \tag{4.5}$$

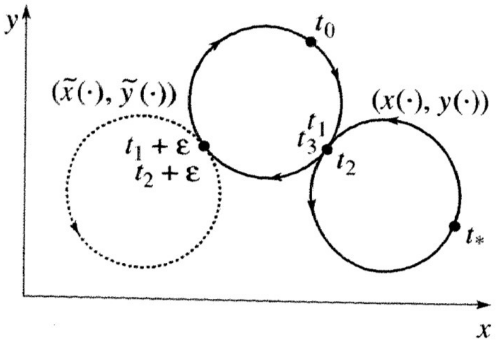

Fig. 5. Explanation of the proof of Lemma 3.

Further,

$$\tilde{x}(\hat{t}) = \tilde{x}(\bar{t}) + \int_{\bar{t}}^{\hat{t}} V\cos\tilde{\varphi}(s)ds$$

$$= \tilde{x}(\bar{t}) - \int_{\hat{t}}^{\bar{t}} V\cos\tilde{\varphi}(\bar{t} + \hat{t} - \tau)d\tau.$$

Using (4.5) and taking into account that $\tilde{x}(\bar{t}) = x(\bar{t})$, we obtain

$$\tilde{x}(\hat{t}) = x(\bar{t}) + \int_{\bar{t}}^{\hat{t}} V\cos\varphi(\tau)d\tau = x(\hat{t}).$$

The equality $\tilde{y}(\hat{t}) = y(\hat{t})$ is proved similarly.

Thus, we have proved that $\tilde{z}(\hat{t}) = z(\hat{t})$; i.e., at the instant $\hat{t}$, the auxiliary motion $\tilde{z}(\cdot)$ has the same phase coordinates as the original motion $z(\cdot)$. Hence, $\tilde{z}(t_*) = z(t_*)$, and, thus, $\tilde{z}(t_*) \in \partial G(t_*)$. Then, it follows that the control $\tilde{u}(\cdot)$ satisfies the maximum principle.

The motion $\tilde{z}(\cdot)$ has no cycles on the interval $[t_0, t_*]$, which follows from the definition of the control $\tilde{u}(\cdot)$ and from the fact that the motion $z(\cdot)$ has no cycles. Then, it follows from assertion (a) of Lemma 1 that the time intervals between the adjacent switching instants for the control $\tilde{u}(\cdot)$ must be identical. However, this is not true. Indeed, consider three adjacent switching instants $\bar{t}_1$, $\bar{t}_2$, and $\hat{t}$. Using (4.2), we find that $t_2 - t_1 > t_1 - \bar{t}$. Then,

$$\bar{t}_2 - \bar{t}_1 = t_2 - t_1 > t_1 - \bar{t} = \hat{t} - \bar{t}_2$$

and we arrive at the contradiction. Hence, $z(t_*) \in \operatorname{int} G(t_*)$.

Lemma 3. Let a motion $z(\cdot)$ on an interval $[t_0, t_*]$ be generated by a piecewise constant control $u(\cdot)$ taking the values ± 1 with three switching instants. Then, $z(t_*) \in \operatorname{int} G(t_*)$.

Proof. Let (x_1, y_1), (x_2, y_2), and (x_3, y_3) be the geometric position points at the switching instants t_1, t_2, and t_3, respectively.

(i) Let us assume first that the points (x_1, y_1), (x_2, y_2), and (x_3, y_3) are all different. Of two inner intervals $[t_1, t_2]$ and $[t_2, t_3]$, we take the lesser one (or either one if they have equal lengths).

Suppose that the interval $[t_2, t_3]$ has been selected. Then, on the interval $[t_1, t_*]$, the motion $z(\cdot)$ satisfies the assumptions of Lemma 2, and, hence, $z(t_*) \in \operatorname{int} G(t_*)$.

If the interval $[t_1, t_2]$ has been selected, then the assumptions of Lemma 2 are fulfilled for the motion $z(\cdot)$ on the interval $[t_0, t_3]$. In this case, $z(t_3) \in \operatorname{int} G(t_3)$, and, hence, $z(t_*) \in \operatorname{int} G(t_*)$.

(ii) Now, let there be coinciding points among (x_1, y_1), (x_2, y_2), and (x_3, y_3). Assume the contrary; i.e., let $z(t_*) \in \partial G(t_*)$. Then, the control $u(\cdot)$ satisfies the maximum principle.

By virtue of assertion (c) of Lemma 1, the motion $z(\cdot)$ contains at least one cycle. Then, it follows from assertion (b) of Lemma 1 that the points (x_1, y_1), (x_2, y_2), and (x_3, y_3) coincide. This means that the motion on the interval $[t_1, t_2]$ comprises one or several cycles following each other in one direction and the motion on the interval $[t_2, t_3]$ comprises one or several cycles following each other in the opposite direction (Fig. 5).

Consider an auxiliary motion $\tilde{z}(\cdot)$ (Fig. 5) obtained through a small variation of the switching instants t_1 and t_2. Let $t_1 + \varepsilon$ and $t_2 + \varepsilon$, where $0 < \varepsilon < 2\pi/\alpha$, be the new switching instants.

The motions $\tilde{z}(\cdot)$ and $z(\cdot)$ coincide on the interval $[t_2 + \varepsilon, t_*]$. Hence, $\tilde{z}(t_*) = z(t_*)$. Therefore, $\tilde{z}(t_*) \in \partial G(t_*)$. Hence, the maximum principle holds. However, the motion $\tilde{z}(\cdot)$ contains cycles, although the geometric position points at the switching instants do not coincide. This contradicts assertion (b) of Lemma 1. Hence, $z(t_*) \in \operatorname{int} G(t_*)$.

Lemma 4. Let a motion $z(\cdot)$ on the interval $[t_0, t_*]$ be generated by a piecewise constant control $u(\cdot)$ taking values $0, \pm 1$ with two switchings. Suppose that the control is equal to zero on only one interval and this interval is either of the two extreme intervals of control constancy. Then, $z(t_*) \in \operatorname{int} G(t_*)$.

Proof. Let, for definiteness, the control $u(\cdot)$ successively take values 0, 1, and –1. Denote by (x_1, y_1) and (x_2, y_2) the points of the geometric position at the switching instants t_1 and t_2.

Assume the contrary. Let $z(t_*) \in \partial G(t_*)$. Then, the control $u(\cdot)$ satisfies the maximum principle.

By virtue of assertion (d) of Lemma 1, the motion $z(\cdot)$ on the interval $[t_1, t_2]$ comprises one or several cycles following each other (Fig. 6). Hence, $(x_1, y_1) = (x_2, y_2)$.

Consider an auxiliary motion $\tilde{z}(\cdot)$ (Fig. 6) with the initial condition $z(t_0)$ generated by the control

$$\tilde{u}(t) = \begin{cases} 0, & t \in [t_0, t_1) \\ -1, & t \in [t_1, t_1 + \varepsilon) \\ 1, & t \in [t_1 + \varepsilon, t_2 + \varepsilon) \\ -1, & t \in [t_2 + \varepsilon, t_*], \end{cases}$$

where ε is a small number, $0 < \varepsilon < t_* - t_2$, and $\varepsilon < 2\pi/\alpha$.

The motions $\tilde{z}(\cdot)$ and $z(\cdot)$ coincide on the interval $[t_2 + \varepsilon, t_*]$. Hence, $\tilde{z}(t_*) = z(t_*)$. Therefore, $\tilde{z}(t_*) \in \partial G(t_*)$. Hence, the control $\tilde{u}(\cdot)$ satisfies the maximum principle. However, the motion $\tilde{z}(\cdot)$ does not contain cycles on the interval $[t_1, t_1 + \varepsilon]$, which contradicts assertion (d) of Lemma 1.

Hence, $z(t_*) \in \operatorname{int} G(t_*)$.

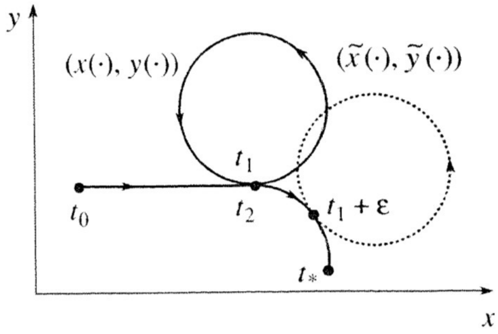

Fig. 6. Explanation of the proof of Lemma 4.

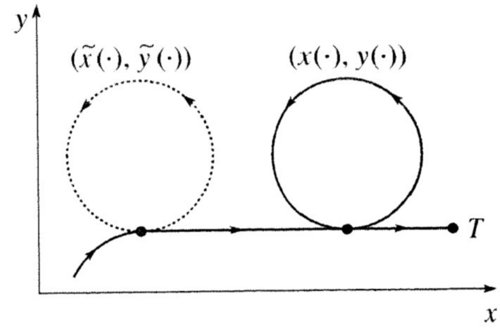

Fig. 7. Explanation of the proof of the theorem.

5. CONTROLS LEADING TO THE BOUNDARY OF THE REACHABILITY SET

Now, we formulate the basic result of the paper.

Theorem. Any boundary point of the reachability set for system (1.1) can be reached by means of a piecewise constant control with no greater than two switchings. In the case of two switchings, it is sufficient to consider six sequences of the control values, namely,

$$\begin{array}{lll} 1)\ 1, 0, 1; & 2)\ -1, 0, 1; & 3)\ 1, 0, -1; \\ 4)\ -1, 0, -1; & 5)\ 1, -1, 1; & 6)\ -1, 1, -1. \end{array} \qquad (5.1)$$

Proof. Assume the contrary. Let there exist a point $\hat{z}$ belonging to the boundary of the reachability set $G(T)$ such that any control that brings the system to this point has three or more switchings. If there are several controls bringing the system to this point, we consider the control with the least number of switchings. Denote this control and the corresponding motion by $u^*(\cdot)$ and $z^*(\cdot)$, respectively.

Consider the motion $z^*(\cdot)$ on the last four intervals of the control constancy. Among these intervals, there may be no greater than two intervals with zero control. The following four variants are possible.

(1) There are no intervals with zero control. Then, by virtue of Lemma 3, we have $z^*(T) \in \operatorname{int} G(T)$. This contradicts the fact that $z^*(T) = \hat{z} \in \partial G(T)$.

(2) There is one interval with zero control. In this case, we can separate three intervals following each other such that the interval with zero control is either the first or last in this triple. It follows from Lemma 4 that $z^*(T) \in \operatorname{int} G(T)$, which contradicts the relation $z^*(T) = \hat{z} \in \partial G(T)$.

(3) The control is zero on two extreme intervals. In this case, similar to case (2) above, using Lemma 4, we find that $z^*(T) \in \operatorname{int} G(T)$ and, thus, arrive at the contradiction.

(4) There are two intervals with zero control, and they are separated by one interval with nonzero control. The control $u^*(\cdot)$ satisfies the maximum principle. Hence, on the strength of assertion (d) of Lemma 1, the interval with nonzero control that separates the intervals with zero control represents one or several cycles following each other (Fig. 7).

We move all cycles from that interval to the initial point of the first linear segment or to the terminal point of the second linear segment. Linking the time coordinates of the linear segments, we obtain an auxiliary motion $\tilde{z}(\cdot)$ that, like the original motion $z^*(\cdot)$, brings the system to the point under consideration at the instant T (Fig. 7). However, the number of control switchings in the auxiliary motion is less than that in the original motion by one, which contradicts the assumption that the selected control $u^*(\cdot)$ has the least number of switchings.

Thus, any point on the boundary of the reachability set $G(T)$ can be reached by means of a piecewise constant control with no greater than two switchings.

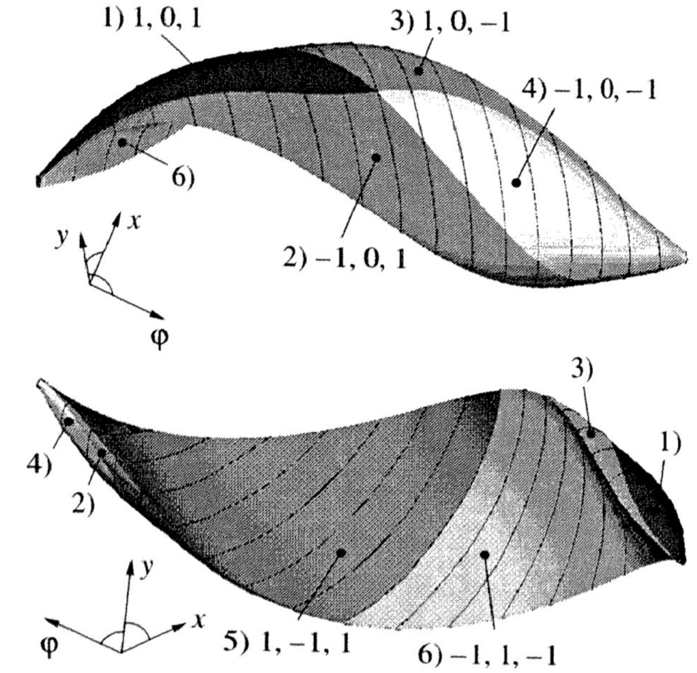

Fig. 8. Structure of the boundary of the reachability set for $T = \pi/\alpha$.

Now, consider the question of possible control sequences. In addition to the six control sequences indicated in the assertion of the theorem, the following six variants are also theoretically possible:

7) 0, 1, −1; 8) 0, −1, 1; 9) 1, −1, 0;
10) −1, 1, 0; 11) 0, 1, 0; 12) 0, −1, 0.

Controls 7 through 10 cannot lead to the boundary of the reachability set on the strength of Lemma 4.

As for the controls 11 and 12, for any motion corresponding to them, the number of switchings can be reduced by one similar to case (4) above. Then, we obtain controls with one switching that lead to the same boundary point. The theorem is proved.

Remark. Note that the above theorem does not completely describe controls that lead to the boundary of the reachability set. Additional easy-to-check conditions are needed that, being applied to the motions corresponding to controls 1–6, would make it possible to eliminate the motions that lead to the interior of the reachability set and keep those that lead to its boundary.

6. NUMERICAL CONSTRUCTION OF THE THREE-DIMENSIONAL REACHABILITY SET

We will apply the theorem from Section 5 to constructing numerically the boundary of the reachability set for system (1.1). The switching instants are considered as parameters.

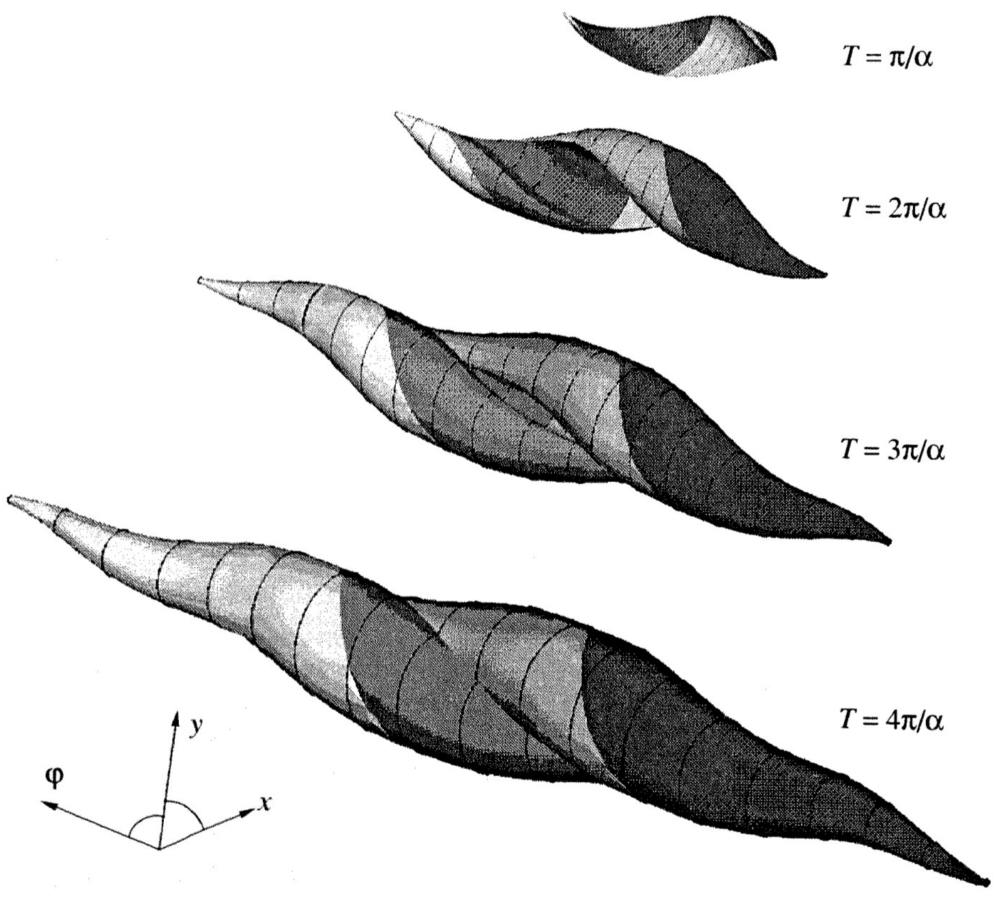

Fig. 9. Evolution of the reachability set.

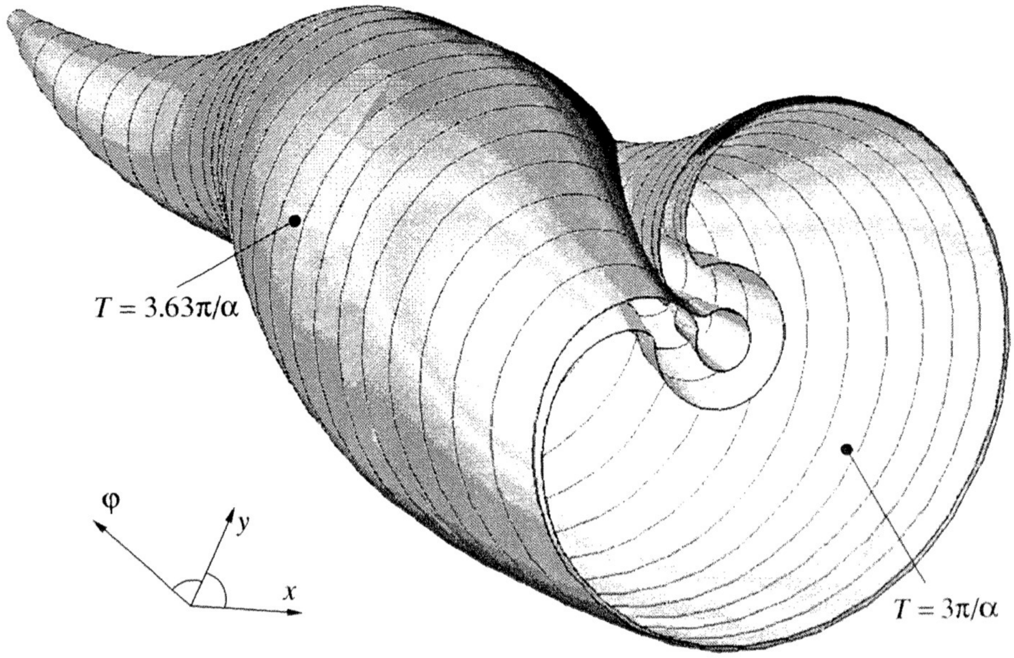

Fig. 10. Violation of the simple connectivity of the reachability set.

We set $t_0 = 0$ and $z_0 = 0$.

To construct the boundary of the reachability set $G(T)$, we sort out all controls with two switchings of form 1–6 from list (5.1). For each control sequence, the parameters t_1 and t_2 are selected from the intervals $[0, T]$ and $[t_1, T]$, respectively. Thus, controls without switchings or with one switching are not excluded from consideration. For a particular control sequence, by allowing the parameters t_1 and t_2 to take values on a sufficiently fine mesh, we obtain a surface in the three-dimensional space of the coordinates x, y, and φ.

Thus, each of the six control sequences in list (5.1) is made to correspond to a surface in the three-dimensional space. The boundary of the reachability set $G(T)$ is composed of fragments of these surfaces. The six surfaces are loaded in a visualization program without any additional processing. By means of this program, we can distinguish the boundary of the reachability set. Some surfaces occur, partially or completely, inside the reachability set. When displaying the boundary, these fragments are not seen.

Figure 8 shows the boundary of the reachability set $G(T)$ at $T = \pi/\alpha$ from two viewpoints. Different fragments of the boundary are depicted by different gradations of gray color. The mesh lines show sections of the reachability set by planes $\varphi = \text{const}$. The control identically equal to zero leads to the point where the fragments 1–4 are joined. Controls with one switching lead to the lines separating the fragments 1, 2; 1, 3; 2, 4; 2, 5; 2, 6; 3, 4; 3, 5; and 3, 6. Any point of the line separating the fragments 5 and 6 can be reached by means of two motions, each with two switchings. The surface along this line is not smooth, but the difference in angles of two surfaces at the joining line is small and cannot be seen in the figure.

The reachability sets $G(T)$ for four instants are shown in Fig. 9 from one viewpoint. One can clearly observe how the structure of the reachability set boundary changes with time: the rear part of the boundary composed of the fragments 5 and 6 is "covered" by the front fragments 1–4.

In the time interval between $T = 3\pi/\alpha$ and $T = 4\pi/\alpha$, there is a instant $T \approx 3.63\pi/\alpha$ starting from which the reachability set is not simply connected during a small interval of time; namely, there appears a cavity that does not belong to the reachability set. Figure 10 shows the genesis of this situation. The figure depicts the sections of two reachability sets $G(T)$ corresponding to the instants $T = 3\pi/\alpha$ and $T = 3.63\pi/\alpha$ by the plane $\varphi = 0$. The set $G(3\pi/\alpha)$ is simply connected, whereas the set $G(3.63\pi/\alpha)$ is not simply connected.

Thus, when T is not too large, the boundary of the three-dimensional reachability set $G(T)$ has a rather simple structure. As T grows, the structure of the boundary becomes more complicated. There exists a small time interval during which the set $G(T)$ is not simply connected.

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