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SPECIAL ASPECTS OF CONVEX HULL CONSTRUCTING IN LINEAR DIFFERENTIAL GAMES OF SMALL DIMENSION

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Abstract: The backward constructions for building the level sets of value function of differential games are discussed. Special aspects are outlined which can be used in fast algorithms. The examples of computer calculations are presented.

Keywords: Differential games, feedback control, numerical methods, terminal control, time-optimal control.

1. INTRODUCTION

In this paper, the linear differential games

$$\dot{x} = A(t)x + B(t)u + C(t)v, \qquad (1)$$

$$x \in R^n, \ u \in P, \ v \in Q, \ \ \gamma(x(T)) \to \min\max$$

with terminal time T and convex payoff function γ, which values are minimized by the first player and maximized by the second one, are considered.

It is known (Krasovskii and Subbotin, 1974, 1988) that such games have the following attractive properties:

- The value function $(t, x) \to V(t, x)$ is convex in x;
- The transformation $y(t) = X(T, t)x(t)$, where $X(T, t)$ is the fundamental Cauchy matrix correspondent to the homogeneous part of the system (1), provides the transition to the equivalent game without the phase vector in the right hand side of the dynamic equation;
- If the payoff function depends on some m coordinates of the phase vector at the terminal time T only, then the transformation $X_m(T, t)x(t)$, where $X_m(T, t)$ are m corresponding rows of the matrix $X(T, t)$, gives the transition to the equivalent m-order differential game.

The paper is devoted to the case when the payoff function depends on two or three coordinates of the phase vector. Such problems are called the games of small dimension.

The backward procedure in coordinates of the equivalent game is described for the approximate constructing the level sets of the value function on the given sequence of time moments. The main difficulty of such constructing corresponds to transition from one time moment to another and consists in finding the convex hull of some piecewise-linear positively-homogeneous function. In the paper, the attention is drawn to characteristics which can be used for obtaining the fast algorithms of constructing the convex hull: before the beginning the convexing process we know the places where the local convexity can be violated. So, the iterative process of constructing the convex hull begins just at such places.

In spite of the fact that in linear differential games with nonfixed terminal time, the level sets of the value function are, as a rule, not convex, the main specific ideas involved into the algorithm of the backward procedure for the games with fixed terminal time can be applied for this case too. At the end of the paper, the examples of constructing the level sets of the value function for the linear time-optimal games in the plane are given.

Numerical methods for solving differential games based on the backward procedures are developed in papers of V.N.Ushakov and his collaborators (see e.g. Ushakov, 1981; Taras'yev, *et al.*, 1988). The results presented in this paper were obtained by the author together with N.D.Botkin, V.L.Turova, M.A.Zarkh.

2. LINEAR DIFFERENTIAL GAMES WITH FIXED TERMINAL TIME

2.1 Backward procedure

Assume that the transfer from the game (1) with the payoff function γ depending on m coordinates of the phase vector to the equivalent game

$$\dot{y} = D(t)u + E(t)v, \tag{2}$$

$$D(t) = X_m(T,t)B(t), \quad E(t) = X_m(T,t)C(t),$$

$$y \in R^m, \quad u \in P, \quad v \in Q, \quad \gamma(y(T))$$

is already done. Let on the interval $[t_*, T]$ the sequence of time moments t_i: $t_N = T, \ldots, t_i = t_{i+1} - \Delta, \ldots, t_0 = t_*$ dividing the interval with the step Δ is given. The interest is in finding the level sets $W_c(t_i) = \{y \in R^m: V(t_i, y) \le c\}$ of the value function V for the given value of parameter c.

Replace the dynamics (2) by the piecewise-constant dynamics

$$\dot{y} = \mathbf{D}(t)u + \mathbf{E}(t)v, \tag{3}$$

$$\mathbf{D}(t) = D(t_i), \quad \mathbf{E}(t) = E(t_i), \quad t \in [t_i, t_{i+1}).$$

Instead of the sets P and Q, let us consider their polyhedral approximations $\mathbf{P}$, $\mathbf{Q}$. Let $\hat{\gamma}$ be the approximating payoff function. For any c, its level set $\mathbf{M}_c = \{y : \hat{\gamma}(y) \le c\}$ is a convex polyhedron.

The approximating game (3) is taken so that, for each step $[t_i, t_{i+1}]$ of the backward procedure, we deal with the game with simple motions, polyhedral convex control constraints and the convex polyhedral target set. Put $\mathbf{W}_c(t_N) = \mathbf{M}_c$, next, find the game solvability set $\mathbf{W}_c(t_{N-1})$, then $\mathbf{W}_c(t_{N-2})$, and so on. As a result, the collection of convex sets is obtained which approximate (Ponomarev and Rozov, 1978; Botkin, 1982) in the Hausdorff metrics the level sets $W_c(t_i)$ of the value function in the game (2).

The set $\mathbf{W}_c(t_i)$ can be represented (Pschenichnyi and Sagaidak, 1970) by the formula

$$\mathbf{W}_c(t_i) = \bigcap_{v \in \mathbf{Q}} (\mathbf{W}_c(t_{i+1}) - \Delta D(t_i)\mathbf{P} - \Delta E(t_i)v)\,.$$

The main peculiarity is the following: the convex sets to be intersected differ each from other with the shear vector only.

Let us formulate the mentioned property in terms of the convexing operation. The support function $l \to \rho(l, \mathbf{W}_c(t_i))$ of the set $\mathbf{W}_c(t_i)$ is the convex hull of the function

$$\varphi(l, t_i) = \rho(l, \mathbf{W}_c(t_{i+1}))$$

$$+\Delta\rho(l, -D(t_i)\mathbf{P}) - \Delta\rho(l, E(t_i)\mathbf{Q}).$$

The function $\varphi(\cdot, t_i)$ is positively-homogeneous and piecewise-linear. The property of local convexity of this function can be violated only at the frontier of the linearity cones of the function $\rho(\cdot, E(t_i)\mathbf{Q})$, i.e. at the frontier of the cones generated by the normals to the big faces of the polyhedron $E(t_i)\mathbf{Q}$ which have just the same vertex.

2.2 Convexing algorithm in two-dimensional case

In the case $m = 2$, the sets in the plane are used. Let us agree to omit the argument t_i in the notation of the function φ. The linearity cones of φ are determined by the normals to the convex polygons $\mathbf{W}_c(t_{i+1}) - D(t_i)\mathbf{P}$, $E(t_i)\mathbf{Q}$. Gathering the outer normals of these sets and ordering them clockwise, the collection L of the vectors is obtained. The collection of values $\varphi(l)$ of the function φ on the vectors $l \in L$ is denoted by Φ. The collections L, Φ describe completely the function φ.

The collection of the normals to $E(t_i)\mathbf{Q}$ ordered clockwise is denoted by S. The collection S is called the collection of "suspicious" vectors. The name is connected with the fact that the function φ is locally convex on the cones which interior does not contain the normals of the set $E(t_i)\mathbf{Q}$. The violation of the local convexity can appear only on the cones which interior contains at least one normal of the polygon $E(t_i)\mathbf{Q}$.

Let $L^{(1)} = L$, $\Phi^{(1)} = \Phi$, $S^{(1)} = S$. The $k+1$ step of the iterative convexing process consists in replacing the collections $L^{(k)}$, $\Phi^{(k)}$ by the collections $L^{(k+1)} \subset L^{(k)}$, $\Phi^{(k)} \subset \Phi^{(k+1)}$. The collection $S^{(k)}$ is also replaced by the new one $S^{(k+1)}$.

Describe now one step of the convexing process. Suppose that the angle between two neighboring vectors from the collection $L^{(k)}$ counted clockwise is less than π. Let $l \to \varphi^{(k)}(l)$ be the piecewise-linear function determined by the collections $L^{(k)}$, $\Phi^{(k)}$. Since $L^{(k)} \subset L^{(k-1)} \subset \ldots \subset L^{(1)}$, $\Phi^{(k)} \subset \Phi^{(k-1)} \subset \ldots \subset \Phi^{(1)}$, then for any vector $\bar{l} \in L^{(k)}$ the value $\varphi^{(k)}(\bar{l})$ is equal to $\varphi(\bar{l})$.

Take some vector $l_* \in S^{(k)}$ and check the local convexity of the function $\varphi^{(k)}$ on the cone generated by the vector l_* and two its neighboring vectors l_- and l_+ selected counterclockwise and clockwise from the collection $L^{(k)}$. In other words, check whether the inequality $l_*'y \le \varphi(l_*)$ is active in the triple of the inequalities

$$l_-'y \le \varphi(l_-), \quad l_*'y \le \varphi(l_*), \quad l_+'y \le \varphi(l_+).$$

If the system of three inequalities is compatible, then (by virtue of the ordering the vectors l_-, l_*, l_+) only the middle one can be inactive.

The algorithm of verification: find the intersection point y_* of the straight lines $l_-'y = \varphi(l_-)$ and $l_*'y = \varphi(l_*)$, and then check the inequality

$l'_+ y_* \le \varphi(l_+)$. If it holds, the local convexity takes place (the middle inequality is active). In the opposite case, the local convexity is absent (the middle inequality is inactive).

In the first case, the vector l_* is taken away from the collection $S^{(k)}$, and the remained set is denoted by $S^{(k+1)}$. Let $L^{(k+1)}=L^{(k)}$, $\Phi^{(k+1)}=\Phi^{(k)}$.

In the second case, two situations are distinguished. Let α be the angle counted clockwise from l_- to l_+:

- $\alpha < \pi$. The vector l_* is taken away from the collection $S^{(k)}$, and, simultaneously, the vectors l_- and l_+ are included into the collection (one of them or even both could be already presented in the collection $S^{(k)}$). Denote the new collection of the "suspicious" vectors by $S^{(k+1)}$. The difference of the new collection $L^{(k+1)}$ from the collection $L^{(k)}$ is that the vector l_* is absent in $L^{(k+1)}$. When processing $\Phi^{(k)}$ to $\Phi^{(k+1)}$, the values $\varphi^{(k)}(l_*) = \varphi(l_*)$ are taken away;
- $\alpha \ge \pi$. It means that the discussed triple of inequalities is incompatible. Thus, the convex hull of the function φ does not exist, i.e. $\mathbf{W}_c(t_i) = \emptyset$. The constructing is ceased.

One step of the convexing algorithm has been described. The algorithm finishes at the step with the number j, when for the first time $S^{(j)} = \emptyset$, i.e. when the collection of the "suspicious" vectors is empty. It means that the function $\varphi^{(j)}$, which corresponds to the collections $L^{(j)}$ and $\Phi^{(j)}$, is locally convex everywhere. Thus, the function $\varphi^{(j)}$ is the convex hull of the function φ. The second variant of termination is the following: the angle α between the vectors l_- and l_* becomes greater or equal to π after rejecting the checked vector l_* from the collection of the "suspicious" vectors at some step. It means that $\mathbf{W}_c(t_i) = \emptyset$.

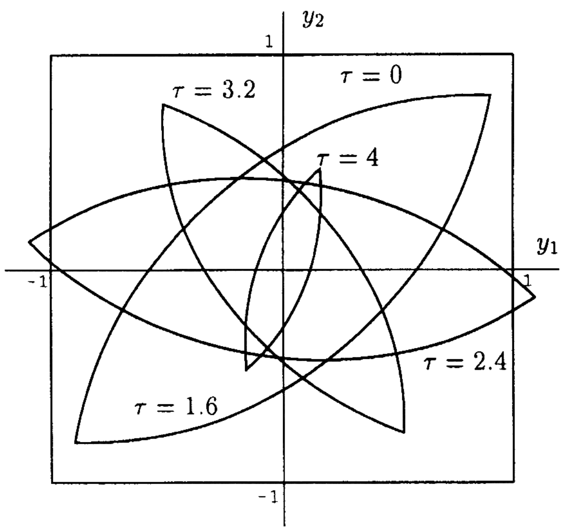

Fig. 1. Level sets of the value function in the differential game (4) for $c = 0.9$.

The algorithm operates very effectively when the number of sides of the polygon $E(t_i)\mathbf{Q}$ is not very large and the step $\Delta = t_{i+1} - t_i$ is not too big. The computer program of the backward constructions which uses the convexing algorithm described above was given in Isakova, *et al.* (1984).

In Fig. 1, the level sets of the value function are presented for the differential game

$$\begin{aligned} \dot{x}_1 &= \quad x_2 + v \\ \dot{x}_2 &= \quad -x_1 + u, \end{aligned} \tag{4}$$

$|\, u \,| \le 1, \;\; |\, v \,| \le 1, \;\; \gamma(x(T)) = \max\{x_1(T), x_2(T)\}.$

The sets are shown in coordinates of the equivalent game. They correspond to $c = 0.9$ and are computed for several reverse time moments $\tau = T - t$.

The other version of the algorithm of processing the set $W_c(t_{i+1})$ to the set $W_c(t_i)$ was proposed in Botkin (1984). In this version, the frontier of the convex polygon $W_c(t_i)$ is determined as a result of intersection of polygonal lines. Each line is constructed via extremal motions emanating from that part of the frontier of the polygon $W_c(t_{i+1})$ which normals lay between two neighboring "suspicious" vectors from the collection S. The algorithm can be modified for solving two-dimensional games which level sets are not necessarily convex: it is possible to use the property of a polygonal line in the plane to be represented piecewise as a "convex" or "concave" curve. Just in this way, V.L.Turova has developed the algorithm for solving the time-optimal games in the plane.

2.3 Three-dimensional problems and the problems of higher dimension

The idea of using *a priory* information about places of violations of the local convexity was realized (Zarkh and Patsko, 1988; Zarkh, 1990a) in the case $m = 3$ under the assumption that the scalar components u_i of the first player's control and scalar components v_j of the second player's control are restricted by the constraints $|u_i| \le \mu_i$, $|v_j| \le \nu_j$. The backward procedures for constructing the level sets in the case $m = 3$ can be used (Subbotin and Taras'yev, 1985) for finding the epigraph of the value function $y \to V(t, y)$, where $y \in R^2$. In Fig. 2, the epigraph of the value function of the game (4) is presented for the reverse time moment $\tau = 2.4$.

Zarkh and Ivanov (1992) developed the algorithm for solving the problems in the case $m > 3$. In Botkin and Ryazantseva (1992), the universal algorithm based on the theory of linear inequalities was described for the case $m \ge 3$.

2.4 The switch surfaces

Having the level sets of the value function constructed, it is possible to find the optimal strategies. In the natural cases, the optimal strategy

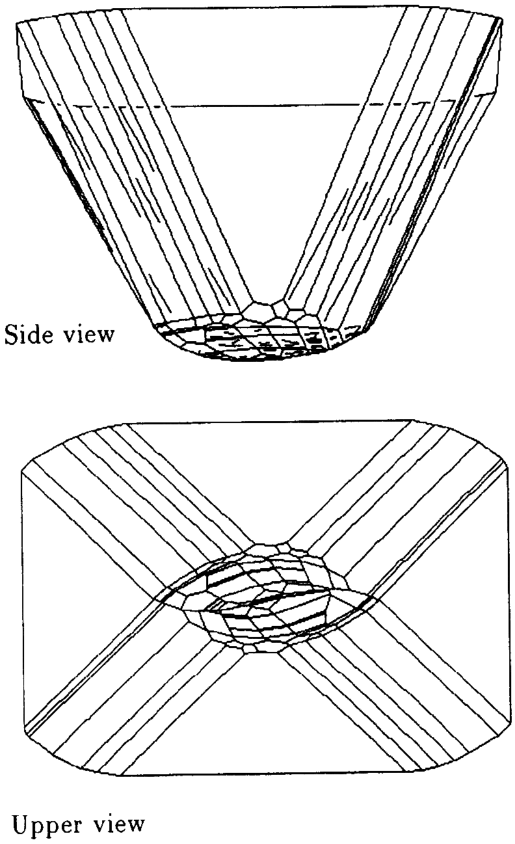

Fig. 2. Epigraph of the value function in the differential game (4), $\tau = 2.4$.

of the first (second) player is given via switch surfaces. The switch surfaces are constructed in the space of variables of the equivalent game. The control is implemented with some step in time.

The switch surface $\Pi^{(1)}(t)$ of the first player for the moment t is the surface dividing the space R^m (or sufficiently big region of the space) onto two parts: the first player's control takes one extremal value in one part, and it takes another extremal value in the other part. It is absolutely clear when the control u is a scalar or when the control u is a vector but its components u_i are restricted by constraints $|u_i| \leq \mu_i$. In the last case, we say about an own switch surface for each component.

The simplest numerical constructions are realized (Botkin and Patsko, 1982) in the case $m = 2$. Here, the switch lines are built. The construction of switch surfaces for $m = 3$ was described in Zarkh (1990a).

The property of stability of the first player's scalar optimal control with respect to the errors of the numerical constructions was justified by Botkin and Patsko (1983). If the control u is a vector and its components are restricted by independent constraints, then the fulfilment of some conditions (connected with possible linking the surfaces corresponding to different components of the control vector) must be required to state strictly that the constructed system of switch surfaces determines the optimal feedback control.

Similarly, the optimal control of the second player can be defined (Zarkh, 1990b) using the switch surfaces. But the property of stability is absent even though in the scalar case.

3. TIME-OPTIMAL GAMES IN THE PLANE

Let consider a conflict-controlled system with linear dynamics and geometrical bounds on controls

$$\dot{x} = Ax + u + v, \qquad (5)$$
$$x \in R^2, \quad u \in P, \quad v \in Q.$$

Here P and Q are convex closed polygons in the plane. The terminal set M (a convex polygon in the plane) is given. The first player governs the control vector u and seeks to minimize the time of attaining M, the aim of the second player governing the control vector v is opposite. The permissible controls are the feedback controls.

It is necessary to build the sets $W(\theta, M)$, $\theta > 0$. Each of them is the set of all initial states x_0 such that the first player guarantees the transition of the state vector to M by the time θ. The set $W(\theta, M)$ is the level set of the value function.

3.1 The main idea of the algorithm

The set $W(\theta, M)$ is formed via step-by-step procedure giving a sequence of embedded sets

$$W(\Delta, M) \subset W(2\Delta, M) \subset W(3\Delta, M)$$
$$\subset \ldots \subset W(i\Delta, M) \subset \ldots \subset W(\theta, M).$$

Here Δ is the step of the backward construction. Let $W(0, M) = M$. The set $W(i\Delta, M)$ consists of all initial points such that the first player brings system (5) into the set $W((i-1)\Delta, M)$ within the time duration Δ.

Before implementing the first step of the backward procedure, we find a usable part (Isaaks, 1965) of the boundary of M. It is defined by the formula

$$\Gamma_0 =$$
$$\mathrm{cl}\{x \in \partial M : \min_{u \in P} \max_{v \in Q} \ell'(Ax + u + v) < 0, \forall \ell \in K_x\}.$$

Here K_x is the cone of normals to the set M at x.

The principal notion of the algorithm is the notion of "front". Suppose the usable part of M consists of one curve only. Let $F_0 = \Gamma_0$. The front F_i (Fig. 3) is the set of all points on the boundary of $W(i\Delta, M)$ for which the minimum guaranteeing time of the attaining the previous set $W((i-1)\Delta, M)$ is equal exactly to Δ. For other points of the boundary of $W(i\Delta, M)$ the optimal time is less than Δ. The line $\partial W(i\Delta, M) \setminus F_i$

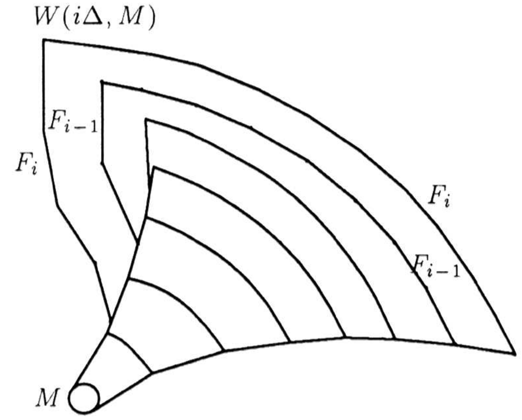

Fig. 3. Construction of the sets $W(i\Delta, M)$.

possesses of the barrier properties. The front F_i is designed using the previous front F_{i-1}.

3.2 Examples of time-optimal problems

1. The canonical example of the time-optimal problem in the theory of optimal control has the following form:

$$\dot{x}_1 = x_2 \qquad \dot{x}_2 = u, \qquad |u| \leq 1.$$

Add the disturbance v to the first equation and consider the differential game

$$\begin{aligned} \dot{x}_1 &= x_2 + v \\ \dot{x}_2 &= u, \qquad |u| \leq 1, \quad |v| \leq 1. \end{aligned} \tag{6}$$

Let M be a small regular octagon with the center at the point $(0,2)$. Put $\Delta = 0.05$. The sets $W(\tau, M)$ for the time instants $\tau = k \cdot 4\Delta$, $k = \overline{1,55}$, are shown in Fig. 4. Denote by a, b the endpoints of the usable part Γ_0 of M. The curves ac and bd formed by the endpoints of fronts are barriers. The value function is discontinuous on these curves and also on the line $\partial M \setminus \Gamma_0$. The line cf

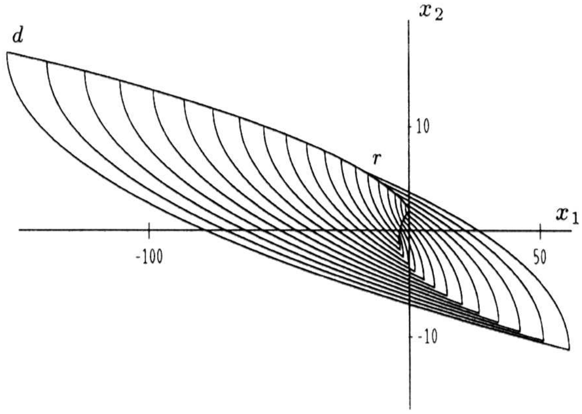

Fig. 5. Differential game (6), solution at $\tau = 20$.

formed by the corners of fronts is the equivocal (Isaaks, 1965) line. The set $W(6.6, M)$ is contoured.The sets $W(\tau, M)$, $\tau = k \cdot 20\Delta$, $k = \overline{1,20}$, are given in Fig. 5. For $\tau > 20$, the front's endpoint r which moves along the upper barrier overtakes another endpoint d. The upper barrier ceases to grow when r coincides with d, and this barrier (as well as the low barrier ac) is extended by an equivocal line. Further, the complicated sequence of low and upper equivocal lines occurs (Patsko, 1972; Filimonov, 1985).

2. Consider the oscillating system

$$\begin{aligned} \dot{x}_1 &= 0.35x_1 + x_2 + v \\ \dot{x}_2 &= -0.8x_1 + u, \end{aligned} \tag{7}$$

$$-2 \leq u \leq 1.5, \quad -6.1 \leq v \leq -4.$$

The terminal set M is a regular octagon with the center at the origin. The level sets $W(\tau, M)$ for $\tau = k \cdot \Delta$, $\Delta = 0.05$, $k = \overline{1,189}$, are given in Fig. 6. Up to $\tau = 5.7$, the front moves between the left and right barrier lines emanating from the set M. The left barrier terminates at $\tau = 5.7$. For $\tau > 5.7$, the front begins to go around this barrier so that

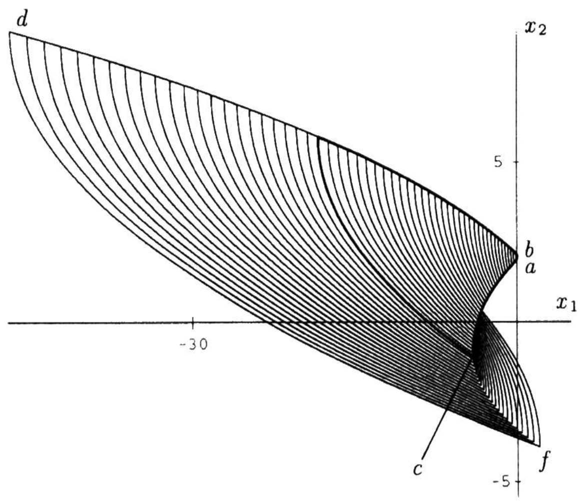

Fig. 4. Differential game (6), solution at $\tau = 11$.

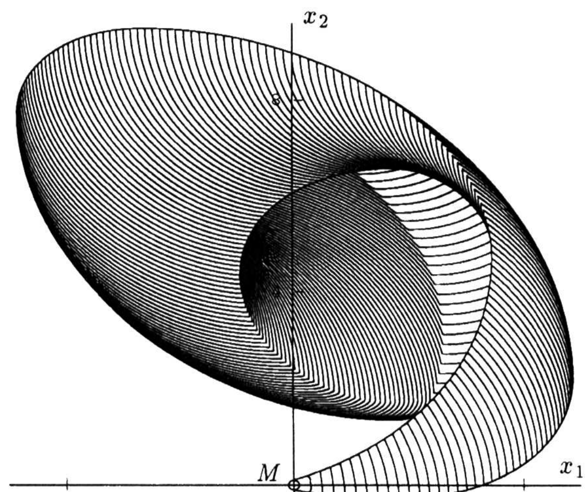

Fig. 6. Level sets of value function in problem (7).

one of its endpoints slides along the outward side of the barrier. At $\tau = 8.15$, the front collides with the initial part of the left barrier from outside. For $\tau > 8.15$, the left and right endpoints of the front move towards each other along the left barrier. The constructions are finished at $\tau = 9.45$. The set, which the fronts fill up by the time $\tau = 9.45$, is the set where optimal guaranteeing time less than infinity. The first player can not guarantee the transferring to M within a finite time from the initial points lying outside this set.

A large number of computer calculated examples for time-optimal problems are given in Turova (1987), Patsko and Turova (1995).

4. CONCLUSION

In this paper, the short review of effective numerical methods for linear differential games of small dimension with fixed terminal time and convex payoff function has been given. The main ideas can be used for time-optimal games in the plane.

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