FITTED MESH METHODS FOR PROBLEMS WITH PARABOLIC BOUNDARY LAYERS

By

J. J. H. Miller
Department of Mathematics, Trinity College Dublin

E. O'Riordan
School of Mathematical Sciences, Dublin City University

G. I. Shishkin
Institute for Mathematics and Mechanics, Russian Academy of Sciences, Ekaterinburg

and

L. P. Shishkina
Scientific Research Institute of Heavy Machine Building, Ekaterinburg

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Abstract

A Dirichlet boundary value problem for a linear parabolic differential equation is studied on a rectangular domain in the $x - t$ plane. The coefficient of the second order space derivative is a small singular perturbation parameter, which gives rise to parabolic boundary layers on the two lateral sides of the rectangle. It is proved that a numerical method, comprising a standard finite difference operator (centred in space, implicit in time) on a fitted piecewise uniform mesh of $N_x \times N_t$ elements condensing in the boundary layers, is uniform with respect to the small parameter, in the sense that its numerical solutions converge in the maximum norm to the exact solution uniformly well for all values of the parameter in the semi-open interval (0,1]. More specifically, it is shown that the errors are bounded in the maximum norm by $C((N_x^{-1} \ln N_x)^2 + N_t^{-1})$, where C is a constant independent not only of N_x and N_t but also of the small parameter. Numerical results are presented, which validate numerically this theoretical result and show that a numerical method consisting of the same finite difference operator on a *uniform* mesh of $N_x \times N_t$ elements is not uniform with respect to the small parameter.

1. Introduction

Boundary layers occur in the solution of singularly perturbed problems when the singular perturbation parameter, which multiplies terms involving the highest derivatives in the differential equation, tends to zero. These boundary layers are neighbourhoods of the boundary of the domain, where the solution has a very steep gradient. Away from any corner of the domain a boundary layer of either regular or parabolic type may occur. A boundary layer is said to be of parabolic type if the characteristics of the reduced equation, corresponding to $\varepsilon = 0$, are parallel to the boundary, and of regular type if these characteristics are not parallel to the boundary. A boundary layer near a corner is said to be of corner type.

Due to the presence of the steep gradients numerical methods using standard finite difference operators on uniform meshes are not adequate for solving problems with boundary layers. Furthermore, it is important that the convergence analysis is in the maximum norm rather than an averaged norm, in order that the singular components of the solution are detected. These considerations lead to the concept of an ε-uniform method, which is a numerical method for solving a singularly perturbed problem having an error estimate in the maximum norm that is independent of the size of the singular perturbation parameter ε.

When regular boundary layers are present it is often possible to obtain an ε-uniform method by constructing an appropriately fitted finite difference operator on a uniform mesh. However, this approach is not possible if a parabolic boundary layer is present. This negative result was first proved in Shishkin [3] (see also Miller et al. [2] for a more detailed proof). The main goal of the present paper is to prove in detail the positive result that for linear parabolic problems having parabolic boundary layers, an ε-uniform method can be constructed using a standard finite difference operator on an appropriately-fitted piecewise uniform mesh condensing in the boundary layers.

A description of the contents of the paper follows. The problem is formulated in §2 after the appropriate Hölder spaces are introduced. The corresponding reduced problem is defined and the parabolic boundary layers are described. The maximum principle for the differential operator is discussed and it is shown that this leads immediately to its ε-uniform stability. Sufficient compatibility conditions on the initial and boundary data to guarantee the existence, uniqueness and appropriate regularity of the solutions to the problem are then presented. In §3 both classical and new sharper ε-uniform bounds in the maximum norm on the derivatives of the solution are discussed. The latter are obtained by means of a new decomposition of the solution, which leads to a deceptively simple proof of the required results. The fitted mesh finite difference method is constructed in §4 and a detailed proof that it is an ε-uniform method is given in §5. In §6 numerical results are reported, which validate the results predicted by the theory, and in fact show that the numerical methods work equally well in practice for a much broader class of problems than the theory predicts. It is also shown that a classical numerical method on a uniform mesh is not ε-uniform for the problem under consideration.

The main theoretical result of this paper, presented in §5, was first stated by one of the authors in [3], which contains only a brief outline of the main points of the proof, and is therefore quite difficult to understand.

The paper ends with §7 which summarizes the main conclusions.

2. Formulation of the problem

To discuss the regularity of the solutions to the time-dependent problems considered here some spaces of functions, Hölder continuous in both x and t, are introduced. To be precise, let $\Omega \subset \mathbb{R}$ and let D be a convex domain in $\overline{\Omega} \times [0, T]$. Suppose that $\lambda \in \mathbb{R}$ satisfies $0 < \lambda \le 1$. Then a function u is said to be Hölder continuous in D of degree λ if, for all (x, t), $(x', t') \in D$,

$$|u(x,t) - u(x',t')| \le C(|x - x'|^2 + |t - t'|)^{\lambda/2}.$$

Note the difference in the metrics used for the space and time variables. The set of all Hölder continuous functions forms a normed linear vector space $C^0_\lambda(D)$ with the norm

$$\|u\|_{\lambda,D} = \|u\|_D + \sup_{(x,t),(x',t')\in D} \frac{|u(x,t) - u(x',t')|}{(|x - x'|^2 + |t - t'|)^{\lambda/2}},$$

where

$$\|u\|_D = \sup_{(x,t)\in D} |u(x,t)|.$$

For each integer $k \ge 1$ the following subspaces $C^k_\lambda(D)$ of $C^0_\lambda(D)$, which are functions having Hölder continuous derivatives, are also introduced

$$C^k_\lambda(D) = \left\{ u\colon \frac{\partial^{i+j} u}{\partial x^i \partial t^j} \in C^0_\lambda(D) \text{ for all non-negative integers } i, j \text{ with } 0 \le i + 2j \le k \right\}.$$

The norm on $C^k_\lambda(D)$ is taken to be

$$\|u\|_{k,\lambda,D} = \max_{0\le i+2j\le k} \left\| \frac{\partial^{i+j} u}{\partial x^i \partial t^j} \right\|_{\lambda,D}.$$

Notice again the difference in the treatment of the space and time derivatives. For $u \in C^k_\lambda(D)$ and $0 \le l \le k$ the following semi-norms are also defined:

$$|u|_{l,\lambda,D} = \max_{i+2j=l} \left\| \frac{\partial^{i+j} u}{\partial x^i \partial t^j} \right\|_{\lambda,D}.$$

It is clear from these definitions that

$$\|u\|_{k,\lambda,D} = \max_{0\le l\le k} |u|_{l,\lambda,D},$$

where the notational convention $\|u\|_{0,\lambda,D} = |u|_{0,\lambda,D} = \|u\|_{\lambda,D}$ is adopted. When the domain is obvious, or of no particular significance, D is usually omitted.

Let $\Omega = (0, 1)$, $D = \Omega \times (0, T]$ and $\Gamma = \Gamma_l \cup \Gamma_b \cup \Gamma_r$, where Γ_l and Γ_r are the left and right sides of the box D and Γ_b is its base. Notice that Γ comprises the base and the two sides of the box, while D includes its lid. The notation $\overline{D} = D \cup \Gamma$ is also used. The problem considered is the following linear parabolic partial differential equation in D with Dirichlet boundary conditions on Γ:

$$(P_\varepsilon) \quad \begin{cases} L_\varepsilon u_\varepsilon(x,t) \equiv -\varepsilon \dfrac{\partial^2 u_\varepsilon(x,t)}{\partial x^2} + b(x,t) u_\varepsilon(x,t) + d(x,t) \dfrac{\partial u_\varepsilon(x,t)}{\partial t} = f(x,t), \\ \text{for } (x,t) \in D, \qquad u_\varepsilon = \varphi \text{ on } \Gamma, \\ \text{where } d(x,t) > \delta > 0 \quad \text{and} \quad b(x,t) \ge \beta \ge 0 \text{ in } \overline{D}. \end{cases}$$

The reduced problem corresponding to (P_ε) is

$$(P_0) \quad \begin{cases} bv_0 + d\dfrac{\partial v_0}{\partial t} = f & \text{in } D, \\ v_0 = \varphi & \text{on } \Gamma_b. \end{cases}$$

It is then clear that the solution of (P_ε) has boundary layers on Γ_l and Γ_r. The characteristics of (P_0) are the vertical lines $x =$ constant, which implies that any boundary layers arising in the solution are of parabolic type.

With the above assumptions on the coefficients d and b, L_ε satisfies the following minimum principle.

Minimum Principle. *Assume that* $b, d \in C^0(\overline{D})$ *and let* $\psi \in C^2(D) \cap C^0(\overline{D})$. *Suppose that* $\psi \geq 0$ *on* Γ. *Then* $L_\varepsilon\psi \geq 0$ *in* D *implies that* $\psi \geq 0$ *in* $\overline{D}$.

The stability of L_ε and an ε-uniform bound for the solution of (P_ε) in the maximum norm is an easy consequence of this.

Theorem 1. *Let* v *be any function in the domain of the differential operator* L_ε *in* (P_ε). *Then*

$$\|v\| \leq (1 + \alpha T)\max\{\|L_\varepsilon v\|, \|v\|_\Gamma\},$$

and any solution u_ε *of* (P_ε) *has the* ε*-uniform upper bound*

$$\|u_\varepsilon\| \leq (1 + \alpha T)\max\{\|f\|, \|\varphi\|\},$$

where $\alpha = \max_{\overline{D}}\{0, (1 - b)/d\} \leq 1/\delta$.

The existence and uniqueness of a solution of (P_ε) can be established under the assumption that the data are Hölder continuous and also satisfy appropriate compatibility conditions at the two corner points of Γ. The latter conditions are now described using the notation

$$\varphi = \begin{cases} \varphi_l & \text{on } \Gamma_l \\ \varphi_r & \text{on } \Gamma_r \\ \varphi_b & \text{on } \Gamma_b \end{cases}$$

to distinguish the boundary data on the different edges of Γ. Note that φ_l and φ_r are functions of only t, while φ_b is a function of only x. Then the required compatibility conditions at the two corners are

$$\varphi_b(0) = \varphi_l(0), \qquad \varphi_b(1) = \varphi_r(0) \tag{1}$$

and

$$\begin{aligned} -\varepsilon\frac{d^2\varphi_b(0)}{dx^2} + b(0,0)\varphi_b(0) + d(0,0)\frac{d\varphi_l(0)}{dt} &= f(0,0), \\ -\varepsilon\frac{d^2\varphi_b(1)}{dx^2} + b(1,0)\varphi_b(1) + d(1,0)\frac{d\varphi_r(0)}{dt} &= f(1,0). \end{aligned} \tag{2}$$

Note that φ must be sufficiently smooth for (2) to make sense, namely $\varphi_l \in C^1(\Gamma_l)$, $\varphi_b \in C^2(\Gamma_b)$, $\varphi_r \in C^1(\Gamma_r)$.

In the remainder of this paper it is assumed, without loss of generality, that problem (P_ε) has homogeneous boundary data, that is $\varphi = 0$. Because the boundary conditions are homogeneous the previous compatibility conditions (1) and (2) simplify to

$$f(0,0) = f(1,0) = 0. \tag{3}$$

The following classical theorem gives sufficient conditions for the existence of a unique solution.

Theorem 2. *Assume that* $\varphi = 0$, *the data* $b, d, f \in C^0_\lambda(\overline{D})$, *and that the compatibility conditions*

$$f(0,0) = f(1,0) = 0$$

are fulfilled. Then (P_ε) *has a unique solution* u_ε *and* $u_\varepsilon \in C^2_\lambda(\overline{D})$.

3. Bounds on the solution and its derivatives

The error estimate for the fitted mesh finite difference method, which will be described below, is proved under the assumption that the solution of (P_ε) is more regular than is guaranteed by the result in Theorem 2. To obtain this greater regularity stronger compatibility conditions are imposed at the two corners of Γ.

The additional compatibility conditions are

$$\left(d\frac{\partial}{\partial t} + \varepsilon\frac{\partial^2}{\partial x^2}\right)\left(\frac{f}{d}\right)(0,0) = 0 \quad \text{and} \quad \left(d\frac{\partial}{\partial t} + \varepsilon\frac{\partial^2}{\partial x^2}\right)\left(\frac{f}{d}\right)(1,0) = 0 \tag{4}$$

Note that these require additional smoothness of f and d. The existence of a smooth solution for the problem with homogeneous boundary conditions is now established in the following theorem.

Theorem 3. *Assume that* $\varphi = 0$, *the data* $b, d, f \in C^2_\lambda(\overline{D})$ *and that the compatibility conditions*

$$f(0,0) = f(1,0) = 0$$

and

$$\left(d\frac{\partial}{\partial t} + \varepsilon\frac{\partial^2}{\partial x^2}\right)\left(\frac{f}{d}\right)(0,0) = \left(d\frac{\partial}{\partial t} + \varepsilon\frac{\partial^2}{\partial x^2}\right)\left(\frac{f}{d}\right)(1,0) = 0$$

are fulfilled. Then (P_ε) *has a unique solution* u_ε *and* $u_\varepsilon \in C^4_\lambda(\overline{D})$. *Furthermore, the derivatives of the solution* u_ε *satisfy, for all non-negative integers* i, j, *such that* $0 \le i + 2j \le 4$,

$$\left\| \frac{\partial^{i+j} u_\varepsilon}{\partial x^i \partial t^j} \right\|_{\overline{D}} \le C\varepsilon^{-i/2},$$

where the constant C *is independent of* ε.

PROOF. The proof of the first part is given in Ladyzhenskaya *et al.* [1, chap. IV, p. 320]. The bounds on the derivatives are obtained as follows. Transforming the variable x to the stretched variable $\tilde{x} = x/\sqrt{\varepsilon}$ the problem (P_ε) is transformed to the problem

$$(\tilde{P}_\varepsilon) \quad \begin{cases} -\dfrac{\partial^2 \tilde{u}}{\partial \tilde{x}^2} + \tilde{b}\tilde{u} + \tilde{d}\dfrac{\partial \tilde{u}}{\partial t} = \tilde{f} & \text{on } \tilde{D}_\varepsilon \\ \tilde{u} = 0 & \text{on } \tilde{\Gamma}_\varepsilon, \end{cases}$$

where $\tilde{D}_\varepsilon = (0, 1/\sqrt{\varepsilon}) \times (0, T]$ and $\tilde{\Gamma}_\varepsilon$ is its boundary analogous to Γ. The differential equation in $(\tilde{P}_\varepsilon)$ is independent of ε. Applying the estimate (10.5) from [1, p. 352] gives, for all non-negative integers i, j such that $0 \leq i + 2j \leq 4$, and all $\tilde{N}_\delta$ in $\tilde{D}_\varepsilon$,

$$\left\| \frac{\partial^{i+j}\tilde{u}}{\partial \tilde{x}^i \partial t^j} \right\|_{\tilde{N}_\delta} \leq C(1 + \|\tilde{u}\|_{\tilde{N}_{2\delta}}).$$

Here the constant C is independent of $\tilde{N}_\delta$ where, for any $\lambda > 0$, $\tilde{N}_\lambda$ is a neighbourhood of diameter λ in $\tilde{D}_\varepsilon$. Returning to the original variable x it follows that

$$\left\| \frac{\partial^{i+j}u_\varepsilon}{\partial x^i \partial t^j} \right\|_{\overline{D}} \leq C\varepsilon^{-i/2}(1 + \|u_\varepsilon\|_{\overline{D}}).$$

The proof is completed by using the bound on u_ε in Theorem 1. ■

The bounds on the derivatives of the solution given in Theorem 3 were derived from classical results. It turns out, however, that they are not adequate for the proof of the ε-uniform error estimate. Stronger bounds on these derivatives are now obtained by a method originally given in [3]. The key step is to decompose the solution u_ε into smooth and singular components.

Let u_ε be the solution of (P_ε) and write

$$u_\varepsilon = v_\varepsilon + w_\varepsilon, \tag{5}$$

where $v_\varepsilon, w_\varepsilon$ are smooth and singular components of u_ε defined in the following way. The smooth component is further decomposed into the sum

$$v_\varepsilon = v_0 + \varepsilon v_1,$$

where v_0, v_1 are defined by

$$\begin{aligned} bv_0 + d\frac{\partial v_0}{\partial t} &= f \text{ in } D, \qquad & v_0 = 0 \text{ on } \Gamma_b, \\ L_\varepsilon v_1 &= \frac{\partial^2 v_0}{\partial x^2} \text{ in } D, \qquad & v_1 = 0 \text{ on } \Gamma. \end{aligned}$$

It is clear that v_0 is the solution of the reduced problem. Furthermore v_ε satisfies

$$L_\varepsilon v_\varepsilon = f \text{ in } D \quad v_\varepsilon = 0 \text{ on } \Gamma_b \quad \text{and} \quad v_\varepsilon = v_0 \text{ on } \Gamma_l \cup \Gamma_r.$$

With v_ε thus defined, it follows that w_ε is determined and that it satisfies

$$L_\varepsilon w_\varepsilon = 0 \text{ in } D \quad w_\varepsilon = 0 \text{ on } \Gamma_b \quad \text{and} \quad w_\varepsilon = -v_0 \text{ on } \Gamma_l \cup \Gamma_r.$$

It is also convenient to write

$$w_\varepsilon = w_l + w_r,$$

where w_l and w_r are defined by

$$\begin{aligned} L_\varepsilon w_l &= 0 \text{ in } D, & w_l &= -v_0 \text{ on } \Gamma_l, & w_l &= 0 \text{ on } \Gamma_b \cup \Gamma_r, \\ L_\varepsilon w_r &= 0 \text{ in } D, & w_r &= -v_0 \text{ on } \Gamma_r, & w_r &= 0 \text{ on } \Gamma_l \cup \Gamma_b. \end{aligned}$$

It is clear that w_l and w_r correspond respectively to the boundary layers on Γ_l and Γ_r. The required non-classical bounds on v_ε and w_ε, and their derivatives, are contained in the following theorem.

Theorem 4. *Consider the problem* (P_ε). *Assume that the data* $b, d \in C^2_\lambda(\overline{D}), f \in C^4_\lambda(\overline{D})$, *and that the compatibility conditions of the previous theorem are fulfilled. Then the reduced solution* v_0 *exists and* $v_0 \in C^4_\lambda(\overline{D})$. *Also, if the additional compatibility conditions*

$$\frac{\partial^2 f(0,0)}{\partial x^2} = \frac{\partial^2 f(1,0)}{\partial x^2} = 0 \tag{6}$$

are fulfilled, then v_1 *exists and* $v_1 \in C^4_\lambda(\overline{D})$. *Moreover, assuming that the further compatibility conditions*

$$\frac{\partial f(0,0)}{\partial t} = \frac{\partial f(1,0)}{\partial t} = 0 \tag{7}$$

are satisfied, it follows that w_ε *exists and* $w_\varepsilon \in C^4_\lambda(\overline{D})$. *Also, for all non-negative integers* i, j, *such that* $0 \le i + 2j \le 4$

$$\left\| \frac{\partial^{i+j} v_\varepsilon}{\partial x^i \partial t^j} \right\|_{\overline{D}} \le C(1 + \varepsilon^{1-i/2}),$$

and for all $(x, t) \in D$,

$$\left| \frac{\partial^{i+j} w_l(x,t)}{\partial x^i \partial t^j} \right| \le C\varepsilon^{-i/2} e^{-x/\sqrt{\varepsilon}}$$

and

$$\left| \frac{\partial^{i+j} w_r(x,t)}{\partial x^i \partial t^j} \right| \le C\varepsilon^{-i/2} e^{-(1-x)/\sqrt{\varepsilon}},$$

where C *is a constant independent of* ε.

PROOF. See [1, chap. 4] for the existence and regularity results. The bounds on the functions and their derivatives are proved as follows.

The reduced solution v_0 is the solution of a first order differential equation and a classical argument leads to the estimate

$$\left\| \frac{\partial^{i+j} v_0}{\partial x^i \partial t^j} \right\|_{\overline{D}} \leq C. \tag{8}$$

Furthermore, the function v_1 is the solution of a problem of a form to which Theorem 3 applies. It follows that

$$\left\| \frac{\partial^{i+j} v_1}{\partial x^i \partial t^j} \right\|_{\overline{D}} \leq C\varepsilon^{-i/2}. \tag{9}$$

Since

$$\frac{\partial^{i+j} v_\varepsilon}{\partial x^i \partial t^j} = \frac{\partial^{i+j} v_0}{\partial x^i \partial t^j} + \varepsilon \frac{\partial^{i+j} v_1}{\partial x^i \partial t^j},$$

the required estimates of the smooth component v_ε and its derivatives follow by using (8) and (9).

The required bounds on w_l and w_r and their derivatives are obtained analogously. The proof is therefore only given for w_l and its derivatives. To bound w_l, define

$$\psi^{\pm}(x,t) = Ce^{-x/\sqrt{\varepsilon}} e^{\alpha t} \pm w_l(x,t).$$

Then, if C is chosen sufficiently large and $\alpha \geq 0$,

$$\begin{aligned} \psi^{\pm}(x,0) &= Ce^{-x/\sqrt{\varepsilon}} \geq 0, \\ \psi^{\pm}(0,t) &= Ce^{\alpha t} \mp v_0 \geq 0, \\ \psi^{\pm}(1,t) &= Ce^{-1/\sqrt{\varepsilon}} e^{\alpha t} \geq 0 \end{aligned}$$

and

$$L_\varepsilon \psi^{\pm}(x,t) = C(b - 1 + \alpha d) e^{-x/\sqrt{\varepsilon}} e^{\alpha t} \geq 0$$

if α is chosen as in Theorem 1 to be $\alpha = \max_{\overline{D}}\{0, (1-b)/d\}$. It follows from the maximum principle that for all $(x,t) \in \overline{D}$

$$|w_l(x,t)| \leq Ce^{-x/\sqrt{\varepsilon}} e^{\alpha t} \leq Ce^{-x/\sqrt{\varepsilon}}$$

as required.

The bounds on the derivatives of w_l are obtained as follows. First, a transformation is made from x to the stretched variable $\tilde{x} = x/\sqrt{\varepsilon}$. Using the variables $(\tilde{x}, t)$ the parameter ε does not appear in the differential equation and so the appropriate results in [1, §4.10] are applicable to its solution $\tilde{w}_l$. Note that the domain of the stretched variable $\tilde{x}$ is clearly $(0, 1/\sqrt{\varepsilon})$. The argument divides into two cases corresponding to the position of $\tilde{x}$. For each neighbourhood $\tilde{N}_\delta$ in $(2, 1/\sqrt{\varepsilon}) \times (0, T]$ from [1, §4.10]

$$\left\| \frac{\partial^{i+j} \tilde{w}_l}{\partial \tilde{x}^i \partial t^j} \right\|_{\tilde{N}_\delta} \leq C \|\tilde{w}_l\|_{\tilde{N}_{2\delta}}$$

and the required bound follows by transforming back to the variable x and using the bound just obtained on w_l.

Likewise, for each neighbourhood $\tilde{N}_\delta$ in $(0, 2] \times (0, T]$ from [1, §4.10].

$$\left\| \frac{\partial^{i+j} \tilde{w}_l}{\partial \tilde{x}^i \partial t^j} \right\|_{\tilde{N}_\delta} \leq C(1 + \|\tilde{w}_l\|_{\tilde{N}_{2\delta}})$$

and the required bound follows by again transforming back to the variable x, using the bound on w_l and noting that for $\tilde{x} \leq 2$, $e^{-x/\sqrt{\varepsilon}} \geq e^{-2} = C$. This completes the proof. ■

4. Formulation of the numerical method

Problem (P_ε) is now discretised using a fitted numerical method composed of a standard finite difference operator on a fitted piecewise uniform mesh. The finite difference operator has a centered difference quotient in space and a backward difference quotient in time. The fitted piecewise uniform mesh is constructed by dividing $\overline{\Omega}$ into three subintervals

$$\overline{\Omega} = \overline{\Omega}_l \cup \overline{\Omega}_c \cup \overline{\Omega}_r,$$

where $\Omega_l = (0, \sigma)$, $\Omega_c = (\sigma, 1 - \sigma)$, $\Omega_r = (1 - \sigma, 1)$, and the fitting factor σ is chosen to be

$$\sigma = \min \left\{ \frac{1}{4}, 2\sqrt{\varepsilon} \ln N_x \right\}, \tag{10}$$

where N_x denotes the number of mesh elements used in the x-direction. The multi-index notation $N = (N_x, N_t)$ is also used, where N_t is the number of mesh elements in the t-direction.

A piecewise uniform mesh $\Omega_\sigma^{N_x}$ on Ω with N_x mesh elements, $N_x \geq 4$, is obtained by putting a uniform mesh with $N_x/4$ mesh elements on both Ω_l and Ω_r and one with $N_x/2$ mesh elements on Ω_c. A uniform mesh Ω^{N_t} with N_t mesh elements is used on $(0, T)$. The fitted piecewise uniform mesh D_σ^N on D is then defined to be the tensor product

$$D_\sigma^N = \Omega_\sigma^{N_x} \times \Omega^{N_t}$$

and its boundary points Γ_σ^N are $\Gamma_\sigma^N = \overline{D}_\sigma^N \cap \Gamma$. We put $\Gamma_{l,\sigma}^N = \Gamma_\sigma^N \cap \Gamma_l$ and $\Gamma_{r,\sigma}^N = \Gamma_\sigma^N \cap \Gamma_r$. Note that whenever $\sigma = 1/4$ the mesh is uniform and on the other hand when $\sigma = 2\sqrt{\varepsilon} \ln N_x$ the mesh is condensing on the edges Γ_l and Γ_r.

The resulting fitted mesh finite difference method for (P_ε) is then

$$(P_\varepsilon^N) \quad \begin{cases} \text{Find a mesh function } U_\varepsilon \text{ such that, on } \Gamma_\sigma^N, \quad U_\varepsilon = 0 \\ \text{and, on } D_\sigma^N, \quad -\varepsilon \delta_x^2 U_\varepsilon + b U_\varepsilon + d D_t^- U_\varepsilon = f. \end{cases}$$

The finite difference operator L_ε^N in (P_ε^N) is

$$L_\varepsilon^N = -\varepsilon \delta_x^2 + bI + dD_t^-,$$

where, for any mesh function $V_{i,j}$,

$$\delta_x^2 V_{i,j} = \frac{(D_x^+ - D_x^-)V_{i,j}}{(x_{i+1} - x_{i-1})/2}$$

with

$$D_x^+ V_{i,j} = \frac{V_{i+1,j} - V_{i,j}}{x_{i+1} - x_i}, \quad D_x^- V_{i,j} = \frac{V_{i,j} - V_{i-1,j}}{x_i - x_{i-1}}$$

and an analogous definition of D_t^-. It satisfies the following well known discrete minimum principle on $\overline{D}_\sigma^N$.

Discrete Minimum Principle. *Assume that the mesh function Ψ satisfies $\Psi \geq 0$ on Γ_σ^N. Then $L_\varepsilon^N \Psi \geq 0$ on D_σ^N implies that $\Psi \geq 0$ at each point of $\overline{D}_\sigma^N$.*

An immediate consequence of the discrete minimum principle is the following ε-uniform stability property of the operator L_ε^N.

Lemma 5. *If Z is any mesh function such that $Z = 0$ at each point of Γ_σ^N, then on $\overline{D}_\sigma^N$*

$$|Z| \leq (1 + \alpha T) \max_{D_\sigma^N} |L_\varepsilon^N Z|.$$

5. Convergence of the numerical solutions

The main result of this paper is contained in the following theorem.

Theorem 6. *Assume that $b, d, f \in C_\lambda^2(\overline{D})$ and that all of the compatibility conditions of the previous theorem hold. Then the fitted mesh finite difference method (P_ε^N) with the standard finite difference operator L_ε^N and the fitted piecewise uniform mesh D_σ^N, condensing on the edges Γ_l and Γ_r, is ε-uniform for the problem (P_ε) provided that the fitting factor σ is chosen according to the formula* (10) *above. Moreover, the solution u_ε of (P_ε) and the solutions U_ε of (P_ε^N) satisfy the following ε-uniform error estimate for all $N_x \geq 4$:*

$$\sup_{0<\varepsilon\leq 1} \|U_\varepsilon - u_\varepsilon\|_{\overline{D}_\sigma^N} \leq C((N_x^{-1} \ln N_x)^2 + N_t^{-1}),$$

where C is a constant independent of N_x, N_t and ε.

Proof. The solution U_ε of (P_ε^N) is decomposed into smooth and singular components in an analogous manner to the decomposition of the solution u_ε of (P_ε). Thus

$$U_\varepsilon = V_\varepsilon + W_\varepsilon,$$

where V_ε is the solution of the inhomogeneous problem

$$L_\varepsilon^N V_\varepsilon = f \ \text{ in } \ D_\sigma^N, \qquad V_\varepsilon = v_\varepsilon \ \text{ on } \ \Gamma_\sigma^N$$

and therefore W_ε must satisfy

$$L_\varepsilon^N W_\varepsilon = 0 \text{ in } D_\sigma^N, \qquad W_\varepsilon = -v_\varepsilon \text{ on } \Gamma_\sigma^N.$$

The error can then be written in the form

$$U_\varepsilon - u_\varepsilon = (V_\varepsilon - v_\varepsilon) + (W_\varepsilon - w_\varepsilon),$$

and so the smooth and singular components of the error can be estimated separately.

The smooth component of the error is estimated as follows by a classical argument. From the differential and difference equations it is easy to see that

$$L_\varepsilon^N(V_\varepsilon - v_\varepsilon) = f - L_\varepsilon^N v_\varepsilon = (L_\varepsilon - L_\varepsilon^N)v_\varepsilon,$$

and so

$$L_\varepsilon^N(V_\varepsilon - v_\varepsilon) = -\varepsilon\left(\frac{\partial^2}{\partial x^2} - \delta_x^2\right)v_\varepsilon + d\left(\frac{\partial}{\partial t} - D_t^-\right)v_\varepsilon.$$

It follows from classical estimates (see, for example [2, p. 21]) that, at each point (x_i, t_j) in D_σ^N,

$$|L_\varepsilon^N(V_\varepsilon - v_\varepsilon)(x_i,t_j)| \le \begin{cases} \frac{\varepsilon}{3}(x_{i+1} - x_{i-1})\|\frac{\partial^3 v_\varepsilon}{\partial x^3}\| + \frac{d(x_i,t_j)}{2}(t_j - t_{j-1})\|\frac{\partial^2 v_\varepsilon}{\partial t^2}\| \\ \qquad \text{if } x_i = \sigma \text{ or } x_i = 1-\sigma \\ \frac{\varepsilon}{12}(x_i - x_{i-1})^2\|\frac{\partial^4 v_\varepsilon}{\partial x^4}\| + \frac{d(x_i,t_j)}{2}(t_j - t_{j-1})\|\frac{\partial^2 v_\varepsilon}{\partial t^2}\| \\ \qquad \text{otherwise.} \end{cases}$$

Using the estimates of the derivatives of v_ε in Theorem 4 then gives

$$|L_\varepsilon^N(V_\varepsilon - v_\varepsilon)(x_i,t_j)| \le \begin{cases} C[\sqrt{\varepsilon}(x_{i+1} - x_{i-1}) + (t_j - t_{j-1})] \\ \qquad \text{if } x_i = \sigma \text{ or } x_i = 1-\sigma \\ C[(x_i - x_{i-1})^2 + (t_j - t_{j-1})] \\ \qquad \text{otherwise.} \end{cases}$$

Since $x_i - x_{i-1} \le 2N_x^{-1}$, $x_{i+1} - x_{i-1} \le 4N_x^{-1}$ and $t_j - t_{j-1} \le N_t^{-1}$, this leads to

$$|L_\varepsilon^N(V_\varepsilon - v_\varepsilon)(x_i,t_j)| \le \begin{cases} C(\sqrt{\varepsilon}N_x^{-1} + N_t^{-1})] \\ \qquad \text{if } x_i = \sigma \text{ or } x_i = 1-\sigma \\ C(N_x^{-2} + N_t^{-1}) \\ \qquad \text{otherwise.} \end{cases}$$

Now introduce the function

$$\varphi(x_i,t_j) = C[\frac{\sigma}{\sqrt{\varepsilon}}N_x^{-2}\theta(x_i) + (1+t_j)N_x^{-2} + t_jN_t^{-1}],$$

where θ is the piecewise linear polynomial

$$\theta(x) = \begin{cases} \dfrac{x}{\sigma} & \text{for } 0 \leq x \leq \sigma, \\ 1 & \text{for } \sigma \leq x \leq 1-\sigma, \\ \dfrac{1-x}{\sigma} & \text{for } 1-\sigma \leq x \leq 1. \end{cases}$$

Then, for all $(x_i, t_j) \in \overline{D}_\sigma^N$,

$$0 \leq \varphi(x_i, t_j) \leq C(N_x^{-2} \ln N_x + N_t^{-1})$$

and also

$$L_\varepsilon^N \varphi(x_i, t_j) \geq \begin{cases} C(\sqrt{\varepsilon} N_x^{-1} + N_x^{-2} + N_t^{-1}) & \text{if } x_i = \sigma \text{ or } x_i = 1-\sigma \\ C(N_x^{-2} + N_t^{-1}) & \text{otherwise,} \end{cases}$$

where the observations that $\sigma/\sqrt{\varepsilon} \leq 2 \ln N$ and

$$L_\varepsilon^N \theta(x_i) = \begin{cases} \dfrac{\varepsilon N_x}{\sigma} + b(x_i) & \text{if } x_i = \sigma \text{ or } x_i = 1-\sigma \\ b(x_i)\theta(x_i) & \text{otherwise} \end{cases}$$

have been used. Introducing the two functions

$$\psi^\pm(x_i, t_j) = \varphi(x_i, t_j) \pm (V_\varepsilon - v_\varepsilon)(x_i, t_j)$$

it follows that at each point $(x_i, t_j) \in D_\sigma^N$

$$L_\varepsilon^N \psi^\pm(x_i, t_j) \geq 0$$

and at each point $(x_i, t_j) \in \Gamma_\sigma^N$

$$\psi^\pm(x_i, t_j) = \varphi(x_i, t_j) \geq 0.$$

Thus, from the discrete minimum principle

$$\psi^\pm(x_i, t_j) \geq 0 \qquad \text{for all } (x_i, t_j) \in \overline{D}_\sigma^N,$$

and so for all $(x_i, t_j) \in \overline{D}_\sigma^N$

$$|(V_\varepsilon - v_\varepsilon)(x_i, t_j)| \leq \varphi(x_i, t_j) \leq C(N_x^{-2} \ln N_x + N_t^{-1}),$$

that is

$$|V_\varepsilon - v_\varepsilon| \leq C(N_x^{-2} \ln N_x + N_t^{-1}). \tag{11}$$

To estimate the singular component of the error, in an analogous way to that for w_ε, the singular component W_ε is written in the form

$$W_\varepsilon = W_l + W_r,$$

where W_l and W_r are defined by

$$L_\varepsilon^N W_l = 0 \ \text{ in } \ D_\sigma^N, \quad W_l = -v_0 \ \text{ on } \ \Gamma_{l,\sigma}^N, \quad W_l = 0 \ \text{ on } \ \Gamma_{b,\sigma}^N \cup \Gamma_{r,\sigma}^N$$

and

$$L_\varepsilon^N W_r = 0 \ \text{ in } \ D_\sigma^N, \quad W_r = -v_0 \ \text{ on } \ \Gamma_{r,\sigma}^N, \quad W_r = 0 \ \text{ on } \ \Gamma_{l,\sigma}^N \cup \Gamma_{b,\sigma}^N.$$

The error can then be written in the form

$$W_\varepsilon - w_\varepsilon = (W_l - w_l) + (W_r - w_r),$$

and the errors $W_l - w_l$ and $W_r - w_r$, associated respectively with the boundary layers on Γ_l and Γ_r, can be estimated separately.

Consider the error $W_l - w_l$. From the differential and difference equations it is easy to see that

$$L_\varepsilon^N(W_l - w_l) = (L_\varepsilon - L_\varepsilon^N)w_l = -\varepsilon\left(\frac{\partial^2}{\partial x^2} - \delta_x^2\right) w_l + d\left(\frac{\partial}{\partial t} - D_t^-\right) w_l. \tag{12}$$

A classical estimate gives

$$|\left(\frac{\partial}{\partial t} - D_t^-\right) w_l(x_i, t_j)| \le \frac{1}{2}(t_j - t_{j-1})\left\|\frac{\partial^2 w_l}{\partial t^2}\right\|. \tag{13}$$

Using the fact that the t-mesh is uniform with $t_j - t_{j-1} = N_t^{-1}$ and the bounds on the t-derivatives of w_l in Theorem 4, it follows that on D_σ^N the second term on the right of (12) satisfies

$$|d\left(\frac{\partial}{\partial t} - D_t^-\right) w_l| \le CN_t^{-1}. \tag{14}$$

To bound the first term on the right of (12) note that from (10) there are just two possibilities. Either $\sigma = 1/4$ or $\sigma = 2\sqrt{\varepsilon}\ln N_x$. In the first case the mesh is uniform and so $x_i - x_{i-1} = N_x^{-1}$. Also $1/4 \le 2\sqrt{\varepsilon}\ln N_x$ and so $\varepsilon^{-1} \le 64(\ln N)^2$. Combining these with a classical estimate and Theorem 4 yields the bound for all $(x_i, t_j) \in D_\sigma^N$

$$|\varepsilon\left(\frac{\partial^2}{\partial x^2} - \delta_x^2\right) w_l(x_i, t_j)| \le C(N_x^{-1}\ln N_x)^2. \tag{15}$$

In the second case the mesh is piecewise uniform and $\sigma = 2\sqrt{\varepsilon}\ln N_x$. The argument now depends on the position of the mesh point x_i in Ω and there are three distinct possibilities.

The first is $x_i \in (0, \sigma)$. Then $x_i - x_{i-1} = \frac{4\sigma}{N_x} = 8\sqrt{\varepsilon}N_x^{-1}\ln N_x$. Combining this with a classical estimate and Theorem 4 leads to the bound (15).

The second is $x_i \in (\sigma, 1)$. Then $x_{i-1} \ge \sigma$ and so $e^{-x_{i-1}/\sqrt{\varepsilon}} \le e^{-\sigma/\sqrt{\varepsilon}} = e^{-2\ln N_x} = N_x^{-2}$. Combining this with a classical estimate and Theorem 4 gives the bound for all $(x_i, t_j) \in D_\sigma^N$

$$|\varepsilon \left(\frac{\partial^2}{\partial x^2} - \delta_x^2 \right) w_l(x_i, t_j)| \leq C N_x^{-2}. \tag{16}$$

The third is $x_i = \sigma$. Then $x_{i-1} = \sigma - 4\sigma/N_x$ and so

$$\begin{aligned} e^{-x_{i-1}/\sqrt{\varepsilon}} &= e^{-\sigma/\sqrt{\varepsilon}} \,.\, e^{4\sigma N_x^{-1}/\sqrt{\varepsilon}} \\ &= e^{-2\ln N_x} \,.\, e^{8N_x^{-1}\ln N_x} \\ &= N_x^{-2}(N_x^{1/N_x})^8 \\ &\leq C N_x^{-2}, \end{aligned}$$

since $\sup_{M\geq 1} M^{1/M} < \infty$. Combining this result with a classical estimate and Theorem 4 again leads to the bound (16), which is a slightly stronger result than (15).

In all cases therefore the first term on the right of (12) satisfies (15). Combining (14) and (15) with (12) yields the estimate for all $(x_i, t_j) \in D_\sigma^N$

$$|L_\varepsilon^N (W_l - w_l)(x_i, t_j)| \leq C((N_x^{-1} \ln N_x)^2 + N_t^{-1}).$$

Using Lemma 5 then gives for all $(x_i, t_j) \in \overline{D}_\sigma^N$

$$|(W_l - w_l)(x_i, t_j)| \leq C((N_x^{-1} \ln N_x)^2 + N_t^{-1}). \tag{17}$$

A completely analogous argument leads to the estimate for the error corresponding to the boundary layer for all $(x_i, t_j) \in \Gamma_r$

$$|(W_r - w_r)(x_i, t_j)| \leq C((N_x^{-1} \ln N_x)^2 + N_t^{-1}). \tag{18}$$

Combining (11), (17) and (18) completes the proof. ■

Let $\overline{U}_\varepsilon$ denote the piecewise bilinear interpolant of the solution U_ε of (P_ε^N) from the mesh D_σ^N to the domain $\overline{D}$. The following theorem, which is easily established by arguments given in [2], shows that this interpolant is also ε-uniform at each point of $\overline{D}$.

Theorem 7. *Assume that the hypotheses of the previous theorem hold and that $\overline{U}_\varepsilon$ is a piecewise bilinear interpolant of the solution U_ε of (P_ε^N). Let u_ε denote the solution of (P_ε). Then, for all $N_x \geq 4$, the following ε-uniform error estimate holds:*

$$\sup_{0<\varepsilon\leq 1} \|\overline{U}_\varepsilon - u_\varepsilon\|_{\overline{D}} \leq C((N_x^{-1} \ln N_x)^2 + N_t^{-1}),$$

where C is a constant independent of N_x, N_t and ε.

Note that if, in the above theorem, the piecewise constant interpolant $\overline{U}_\varepsilon$ of the exact solution u_ε had been used then the weaker ε-uniform error estimate

$$\sup_{0<\varepsilon\leq 1} \|\overline{U}_\varepsilon - u_\varepsilon\|_{\overline{D}} \leq C(N_x^{-1}\ln N_x + N_t^{-1})$$

would have been determined.

In [4], the author deals with a generalisation of the problem (P_ε) to n space dimensions. Using the obvious generalisation of the piecewise-uniform mesh given in this paper and assuming sufficient compatibility and sufficient smoothness, so that only parabolic boundary layers occur in the solution, it is shown that

$$\sup_{0<\varepsilon\leq 1} \|U_\varepsilon - u_\varepsilon\|_{D^N} \leq C[(N_x^{-1}\ln N_x)^2 + N_t^{-1}],$$

where C is independent of N and ε.

6. Numerical results

Numerical results are presented in this section for the problem with the data $T = 1$, $b(x,t) \equiv 0$, $d(x,t) \equiv 1$, $f(x,t) \equiv 0$, $\varphi(x,0) \equiv 0$, $\varphi(0,t) = t$ and $\varphi(1,t) = (t + 1/2\varepsilon)\text{erfc}\,(\frac{1}{2\sqrt{\varepsilon t}}) - \sqrt{\frac{t}{\pi\varepsilon}}e^{-1/4\varepsilon t}$. The exact solution of this problem is

$$u_\varepsilon(x,t) = (t + \frac{x^2}{2\varepsilon})erfc(\frac{x}{2\sqrt{\varepsilon t}}) - \sqrt{\frac{t}{\pi\varepsilon}}xe^{-x^2/4\varepsilon t}.$$

It is clear that there is a parabolic boundary layer in a neighbourhood of Γ_l, but because of the boundary values there is no boundary layer on Γ_r.

It is easy to verify that not all of the compatibility hypotheses of Theorem 6 are fulfilled by the data of this problem. This means that this problem does not belong to the restricted class of problems covered by Theorem 6. Nevertheless, it is seen below that the numerical behaviour of the appropriate fitted-mesh finite difference method is ε-uniform and it follows that in practice the numerical method is ε-uniform for a wider class of problems than is covered by Theorem 6.

In what follows the problem is solved using numerical methods (P_ε^N) comprising standard finite difference operators (centred in space, implicit in time) on either uniform meshes with $N_x \times N_t$ elements or fitted meshes with $N_x \times N_t$ elements. The fitted meshes used in these computations are of the form described in §4, and so they condense on both Γ_l and Γ_r. But because there is no boundary layer on Γ_r, there is no need for the mesh to condense on Γ_r. This means that equally good numerical results could have been obtained for this problem using a mesh condensing on Γ_l alone and therefore requiring fewer mesh points. The reasons for not removing the mesh condensation on Γ_r was because the available code was written for the more general case and the optimal mesh was not investigated. In the remainder of this section it is assumed that $N_x = N_t = N$.

The errors $E(\varepsilon, N)$ in the numerical solutions using uniform meshes with $N = 4, 16, 64, 256$, and 1024 and values of ε from 1 to 2^{-24} are presented in Table 1.

The last row of the table contains the maximum error, $E(N) = \max_\varepsilon E(\varepsilon, N)$,

TABLE 1—Table of errors $E(\varepsilon, N)$ using classical uniform meshes.

$\varepsilon \setminus N$	4	16	64	256	1024
1.0	1.630e-02	6.144e-03	1.780e-03	4.651e-04	1.176e-04
2^{-4}	4.374e-02	8.624e-03	1.960e-03	4.769e-04	1.184e-04
2^{-8}	3.601e-02	2.558e-02	3.131e-03	5.507e-04	2.484e-04
2^{-12}	2.432e-03	3.095e-02	2.061e-02	1.728e-03	2.444e-04
2^{-16}	1.526e-04	2.069e-03	2.966e-02	1.934e-02	1.376e-03
2^{-20}	9.537e-06	1.297e-04	1.978e-03	2.934e-02	1.902e-02
2^{-24}	5.960e-07	8.106e-06	1.240e-04	1.956e-03	2.926e-02
$E(N)$	4.374e-02	3.095e-02	2.966e-02	2.934e-02	2.926e-02

TABLE 2—Table of errors $E(\varepsilon, N)$ using fitted piecewise uniform meshes.

$\varepsilon \setminus N$	4	16	64	256	1024
1	1.630e-02	6.144e-03	1.780e-03	4.651e-04	1.176e-04
2^{-4}	4.374e-02	8.624e-03	1.960e-03	4.769e-04	1.184e-04
2^{-8}	3.976e-02	2.558e-02	3.131e-03	5.507e-04	2.484e-04
2^{-12}	4.494e-04	4.156e-02	7.214e-03	1.077e-03	2.478e-04
2^{-16}	9.440e-03	4.156e-02	7.214e-03	1.077e-03	2.478e-04
2^{-20}	1.207e-02	4.156e-02	7.214e-03	1.077e-03	2.478e-04
2^{-24}	1.273e-02	4.156e-02	7.214e-03	1.077e-03	2.478e-04
$E(N)$	4.374e-02	4.156e-02	7.214e-03	1.077e-03	2.478e-04

occurring in the rows above it. Since these maxima occur along a diagonal of the table, and do not decrease significantly as N increases, it is clear that there is a persistent maximum error of about 3% no matter how large N is. This shows numerically that this numerical method is not ε-uniform. Another feature of this behaviour is that when a value of ε is chosen that is below the diagonal, then the error grows with increasing N until the diagonal is reached. This behaviour is not in accord with the properties expected of a satisfactory numerical method.

On the other hand the analogous results on the appropriate fitted meshes are presented in Table 2.

In this table, for all $N \geq 16$, the maxima of the columns occur in the row corresponding to $\varepsilon = 2^{-12}$ and these maxima decrease rapidly as N increases. This behaviour is in complete agreement with the theoretical result in Theorem 6. Note that with this ε-uniform method, when $N = 64$, the maximum error in that column is less than 1%, which cannot be achieved for any value of N using a uniform mesh.

Finally, while Theorem 6 reveals nothing about the convergence of the computed

TABLE 3—Table of errors $Q(\varepsilon, N)$ using classical uniform meshes.

$\varepsilon \setminus N$	4	16	64	256	1024
1	1.279e-01	3.269e-02	8.217e-03	2.057e-03	4.496e-03
2^{-4}	4.516e-01	1.293e-01	3.317e-02	8.346e-03	5.904e-03
2^{-8}	8.876e-01	4.332e-01	1.228e-01	3.152e-02	1.086e-02
2^{-12}	1.066e+00	8.863e-01	4.282e-01	1.211e-01	3.111e-02
2^{-16}	1.113e+00	1.066e+00	8.860e-01	4.270e-01	1.207e-01
2^{-20}	1.124e+00	1.113e+00	1.066e+00	8.859e-01	4.267e-01
2^{-24}	1.127e+00	1.124e+00	1.113e+00	1.066e+00	8.856e-01
$Q(N)$	1.127e+00	1.124e+00	1.113e+00	1.066e+00	8.859e-01

TABLE 4—Table of errors $Q(\varepsilon, N)$ using fitted piecewise uniform meshes.

$\varepsilon \setminus N$	4	16	64	256	1024
1	1.279e-01	3.269e-02	8.217e-03	2.057e-03	4.496e-03
2^{-4}	4.516e-01	1.293e-01	3.317e-02	8.346e-03	5.904e-03
2^{-8}	7.867e-01	4.332e-01	1.228e-01	3.152e-02	1.086e-02
2^{-12}	7.727e-01	5.505e-01	2.428e-01	8.507e-02	2.701e-02
2^{-16}	7.690e-01	5.505e-01	2.428e-01	8.507e-02	2.701e-02
2^{-20}	7.680e-01	5.505e-01	2.428e-01	8.507e-02	2.701e-02
2^{-24}	7.678e-01	5.505e-01	2.428e-01	8.507e-02	2.701e-02
$Q(N)$	7.867e-01	5.505e-01	2.428e-01	8.507e-02	2.701e-02

normalised flux to its exact value

$$P_\varepsilon(x, t) = \sqrt{\varepsilon}\frac{\partial u_\varepsilon(x, t)}{\partial x},$$

the following two tables show experimentally that, if the computed normalised flux is defined by

$$P_\varepsilon^N(x, t) = \sqrt{\varepsilon} D_x^+ U_\varepsilon(x, t),$$

then its values, for example on the boundary, converge ε-uniformly in the maximum norm to the correct values using fitted meshes, while the convergence is not ε-uniform using uniform meshes. In Tables 3 and 4

$$Q(\varepsilon, N) = \max_{0 \le t \le T} |P_\varepsilon(0, t) - P_\varepsilon^N(0, t)|$$

and

$$Q(N) = \max_\varepsilon Q(\varepsilon, N).$$

7. Conclusions

A singularly perturbed Dirichlet boundary value problem for a linear parabolic differential equation having parabolic boundary layers was formulated. A fitted mesh finite difference method was constructed and was proved to be an ε-uniform method for this problem. Numerical results were presented, which numerically validate this theoretical result and show that a method using the same finite difference operator on a uniform mesh is not an ε-uniform method.

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