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An alternating direction scheme on a nonuniform mesh for reaction–diffusion parabolic problems

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In this paper we develop a numerical method for two-dimensional time-dependent reaction–diffusion problems. This method, which can immediately be generalized to higher dimensions, is shown to be uniformly convergent with respect to the diffusion parameter.

Keywords: Singular perturbation; alternating direction; Shishkin mesh; uniform convergence.

1. Introduction

In this paper we consider reaction–diffusion problems of the type

$$\begin{cases} \dfrac{\partial u}{\partial t} - \varepsilon \Delta u + k(x, y)u = f(x, y, t), \\ \qquad (x, y, t) \in D \equiv \Omega \times (0, T] \equiv (0, 1)^2 \times (0, T], \\ u(x, y, 0) = u_0(x, y), \qquad (x, y) \in \Omega, \\ u(0, y, t) = u(1, y, t) = 0, \qquad (y, t) \in [0, 1] \times [0, T], \\ u(x, 0, t) = u(x, 1, t) = 0, \qquad (x, t) \in [0, 1] \times [0, T], \end{cases} \tag{1.1}$$

where $0 < \varepsilon$ is the diffusion parameter and $k(x, y) > \underline{k} > 0$. In the singularly perturbed case, when the diffusion parameter ε is small with respect to the reaction coefficient k, the

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solutions of these problems may have a multiscale character, presenting rapid variations in some narrow regions close to the boundary of the domain Ω (parabolic boundary layers). In this case, uniformly convergent methods, i.e., methods in which the rate of convergence and the error constant of the method are independent of the parameter ε, are of great interest. For this type of problem, numerical methods based on exponential fitting techniques are not appropriate for obtaining uniformly convergent methods (see Shishkin, 1990, 1992). To derive uniformly convergent methods using classical discretizations (the central finite difference scheme or standard finite element methods), it is possible to use some special types of nonuniform meshes, introduced by G. I. Shishkin. These kinds of piecewise uniform meshes are defined by taking into account the type and the localization of boundary layers in the problem.

The analysis of uniform convergence with respect to ε of numerical methods based on Shishkin meshes is a subject of increasing interest. For an introduction see the book by Miller *et al.* (1996) and the references given therein, and the papers of Sun & Stynes (1995a, b), Clavero *et al.* (1998), and Hegarty *et al.* (1995), which include many numerical computations for different problems using these meshes. For parabolic problems, we refer to the papers of Hemker *et al.* (1997) and Farrell *et al.* (1996a, b, c).

In this work we propose the use of alternating direction techniques to discretize the time variable. Thus, for the spatial discretizations we can work locally on one-dimensional boundary value problems, of the form

$$\begin{aligned} &-\varepsilon \Delta t w'' + (1 + \Delta t k) w = v + \Delta t g, \qquad \text{on } (0, 1),\\ &\quad w(0) = 0, \quad w(1) = 0, \end{aligned}$$

where v and g are known functions. These problems are discretized by a central finite difference scheme, on appropriate nonuniform meshes. In Section 2 we show a set of optimal bounds for the derivatives of the solution of (1.1). In Section 3 we prove the uniform convergence (with respect to both Δt and ε) of the time semidiscretization. This result, together with the uniform convergence analysis carried out in Section 4 for the spatial discretization, proves uniform convergence of the totally discrete method. Finally, in Section 5 we show some numerical examples, which confirm the theoretical results previously obtained.

Throughout the paper C will denote a generic positive constant independent of ε and the mesh parameters.

2. The continuous problem

In order to perform the convergence analysis in the maximum norm, we will first suppose enough smoothness and compatibility conditions on the data (f, k, u_0), ensuring continuity for the solution of (1.1) and its derivatives up to a certain order in the domain $\overline{D}$. The maximum order is determined by the Taylor expansions used in the space and time consistency analysis. For instance, if $u_0 \in \mathrm{C}^0(\overline{\Omega})$, $f, k \in \mathrm{C}^0(\overline{D})$ and the compatibility condition $u_0(x, y) = 0$ in $\partial\Omega$ holds, then $u \in \mathrm{C}^0(\overline{D})$. By differentiating problem (1.1) with respect to the variable t as far as needed, recurrent applications of the above property can be used to ensure continuity of the derivatives of $u(x, y, t)$. For example, in the first step, $v \equiv \partial u/\partial t$ is the solution to

$$\begin{cases} \dfrac{\partial v}{\partial t} - \varepsilon \Delta v + kv = \dfrac{\partial f}{\partial t}, & \text{in } \Omega \times (0, T], \\ v(x, y, 0) = \varepsilon \Delta u_0 - k u_0 + f(x, y, 0), & (x, y) \in \Omega, \\ v(0, y, t) = v(1, y, t) = 0, & (y, t) \in [0, 1] \times [0, T], \\ v(x, 0, t) = v(x, 1, t) = 0, & (x, t) \in [0, 1] \times [0, T], \end{cases} \tag{2.1}$$

and, in the same way, we can give sufficient conditions for $v \in C^0(\overline{D})$.

Notice that $v, f, k \in C^0(\overline{D})$ implies $u \in C^{2,1}(\overline{D})$. In this situation, we also have

$$f(0, 0, t) = f(0, 1, t) = f(1, 0, t) = f(1, 1, t) = 0, \tag{2.2}$$

since $0 = v(\overline{x}, \overline{y}, t) - \varepsilon \Delta u(\overline{x}, \overline{y}, t) + k\, u(\overline{x}, \overline{y}, t) = f(\overline{x}, \overline{y}, t)$, with $\overline{x},\ \overline{y} \in \{0, 1\}$.

In the analysis of the uniform convergence of the discretization method, a good knowledge of the behaviour with respect to ε of the solution of (1.1) and its derivatives will be needed. Let us assume that the data are smooth and compatible enough so that the solution of (1.1) belongs to $C^{4+\lambda, \frac{4+\lambda}{2}}(\overline{D})$ with $\lambda > 0$. In Shishkin (1990, 1992) a decomposition of the solution of (1.1) of the form $u = U + V$, where U and V are the regular and the singular part of u respectively, was proven as follows.

The function U is taken as $U = u^*|_{\overline{D}}$, where u^* is the solution of the initial boundary value problem

$$\begin{cases} \dfrac{\partial u^*}{\partial t} - \varepsilon \Delta u^* + k^* u^* = f^*, & \text{in } \Omega^* \times (0, T], \\ u^*(x, y, 0) = u_0^*, & (x, y) \in \Omega^*, \\ u^*(x, y, t) = \text{smooth function}, & (x, y, t) \in \partial\Omega^* \times [0, T], \end{cases} \tag{2.3}$$

where Ω^* is a smooth extension of Ω, and u_0^*, f^*, k^* are also smooth prolongations of the functions $u_0,\ f,\ k$ to Ω^*.

The function V is the solution of

$$\begin{cases} \dfrac{\partial V}{\partial t} - \varepsilon \Delta V + kV = 0, & \text{in } \Omega \times (0, T], \\ V(x, y, 0) = 0, & (x, y) \in \Omega, \\ V(x, y, t) = -U(x, y, t), & (x, y, t) \in \partial\Omega \times [0, T]. \end{cases} \tag{2.4}$$

Under the hypotheses made for the data of problem (1.1), U satisfies

$$|\partial^\alpha U| \leqslant C, \quad \forall (x, y, t) \in \overline{D}, \tag{2.5}$$

with $\alpha = (\alpha_1, \alpha_2, \alpha_3),\ \alpha_1 + \alpha_2 + 2\alpha_3 \leqslant 4$.

To obtain appropriate bounds for V, we decompose it in the form

$$V = \sum_{i=1}^{4} V_i + \sum_{i=1}^{4} \overline{V}_i, \tag{2.6}$$

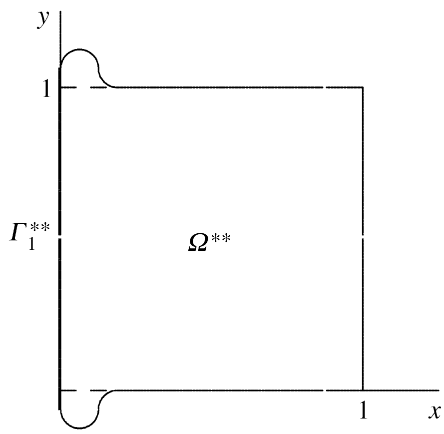

FIG. 1.

where V_i are essentially one-dimensional boundary layer functions of parabolic type in some neighbourhoods of the sides $x = 0, y = 0, x = 1, y = 1$, respectively, and $\overline{V}_i (i = 1, \ldots, 4)$ are respective corner boundary layer functions in some neighbourhoods of the corners $(0, 0), (1, 0), (1, 1), (0, 1)$.

Let Ω^{**} be a smooth extension of Ω near the corners $(0, 1)$ and $(0,0)$ (see Fig. 1), Γ_1^{**} be an extension of the boundary side $x = 0$ beyond the points $(0,0)$ and $(0,1)$, and $\Gamma_2^{**} = \Gamma^{**} \backslash \Gamma_1^{**}$ with $\Gamma^{**} \equiv \partial\Omega^{**}$. Let U^{**} be a smooth and compatible extension of $U(0, y, t)$ to Γ_1^{**} and k^{**} be a smooth prolongation of k to Ω^{**}. The function V_1 (and likewise V_2, V_3, V_4) can be obtained as a restriction to $\overline{D}$ of the solution of

$$\begin{cases} \dfrac{\partial V_1^{**}}{\partial t} - \varepsilon \Delta V_1^{**} + k^{**} V_1^{**} = 0, & \text{in } \Omega^{**} \times (0, T], \\ V_1^{**}(x, y, 0) = 0, & (x, y) \in \Omega^{**}, \\ V_1^{**}(x, y, t) = -U^{**}, & (x, y, t) \in \Gamma_1^{**} \times [0, T], \\ V_1^{**}(x, y, t) = 0, & (x, y, t) \in \Gamma_2^{**} \times [0, T]. \end{cases} \tag{2.7}$$

The function $\overline{V}_1$ is the solution of

$$\begin{cases} \dfrac{\partial \overline{V}_1}{\partial t} - \varepsilon \Delta \overline{V}_1 + k \overline{V}_1 = 0, & \text{in } \Omega \times (0, T], \\ \overline{V}_1(x, y, 0) = 0, & (x, y) \in \Omega, \\ \overline{V}_1(x, y, t) = -(U + V_1 + V_2), & (x, y, t) \in \partial\Omega \times [0, T]. \end{cases} \tag{2.8}$$

Likewise we can define $\overline{V}_2$, $\overline{V}_3$ and $\overline{V}_4$.

Using the above decomposition of V, according to the results of Shishkin (1990, 1992), the following bounds are obtained

$$|\partial^\alpha V_1(x, y, t)| \leqslant C\varepsilon^{-\alpha_1/2} \exp\left(-\frac{\sqrt{\underline{k}}x}{\sqrt{\varepsilon}}\right), \tag{2.9}$$

$$|\partial^\alpha V_3(x, y, t)| \leqslant C\varepsilon^{-\alpha_1/2} \exp\left(-\frac{\sqrt{\underline{k}}(1-x)}{\sqrt{\varepsilon}}\right), \tag{2.10}$$

$$|\partial^\alpha V_2(x, y, t)| \leqslant C\varepsilon^{-\alpha_2/2} \exp\left(-\frac{\sqrt{\underline{k}}y}{\sqrt{\varepsilon}}\right), \tag{2.11}$$

$$|\partial^\alpha V_4(x, y, t)| \leqslant C\varepsilon^{-\alpha_2/2} \exp\left(-\frac{\sqrt{\underline{k}}(1-y)}{\sqrt{\varepsilon}}\right), \tag{2.12}$$

$$|\partial^\alpha \overline{V}_1(x, y, t)| \leqslant C\varepsilon^{-(\alpha_1+\alpha_2)/2} \min\left\{\exp\left(-\frac{\sqrt{\underline{k}}x}{\sqrt{\varepsilon}}\right), \exp\left(-\frac{\sqrt{\underline{k}}y}{\sqrt{\varepsilon}}\right)\right\}, \tag{2.13}$$

$$|\partial^\alpha \overline{V}_2(x, y, t)| \leqslant C\varepsilon^{-(\alpha_1+\alpha_2)/2} \min\left\{\exp\left(-\frac{\sqrt{\underline{k}}(1-x)}{\sqrt{\varepsilon}}\right), \exp\left(-\frac{\sqrt{\underline{k}}y}{\sqrt{\varepsilon}}\right)\right\}, \tag{2.14}$$

$$|\partial^\alpha \overline{V}_3(x, y, t)| \leqslant C\varepsilon^{-(\alpha_1+\alpha_2)/2} \min\left\{\exp\left(-\frac{\sqrt{\underline{k}}(1-x)}{\sqrt{\varepsilon}}\right), \exp\left(-\frac{\sqrt{\underline{k}}(1-y)}{\sqrt{\varepsilon}}\right)\right\}, \tag{2.15}$$

$$|\partial^\alpha \overline{V}_4(x, y, t)| \leqslant C\varepsilon^{-(\alpha_1+\alpha_2)/2} \min\left\{\exp\left(-\frac{\sqrt{\underline{k}}x}{\sqrt{\varepsilon}}\right), \exp\left(-\frac{\sqrt{\underline{k}}(1-y)}{\sqrt{\varepsilon}}\right)\right\}, \tag{2.16}$$

for any $(x, y, t) \in \overline{D}$, with $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, $\alpha_1 + \alpha_2 + 2\alpha_3 \leqslant 4$.

3. The time semidiscretization

Let $k_1(x, y)$, $k_2(x, y)$ be smooth functions satisfying $k_i(x, y) > \gamma^2 > 0$ for $i = 1, 2$ and $k_1 + k_2 = k$. Consider then the following splitting of the spatial differential operator into two operators

$$L_{x,\varepsilon} \equiv -\varepsilon\frac{\partial^2}{\partial x^2} + k_1, \quad L_{y,\varepsilon} \equiv -\varepsilon\frac{\partial^2}{\partial y^2} + k_2. \tag{3.1}$$

Taking into account hypotheses (2.2), we can obtain a decomposition of the source term

$$\begin{aligned} &f = f_1 + f_2, \\ &f_1(x, 0, t) = f_1(x, 1, t) = 0, \quad f_2(0, y, t) = f_2(1, y, t) = 0. \end{aligned} \tag{3.2}$$

The time semidiscretization is carried out by the following alternating direction scheme (see Jorge & Lisbona, 1994):

$$u^0 = u_0(x, y), \tag{3.3}$$

$$(I + \Delta t L_{x,\varepsilon})u^{n+\frac{1}{2}} = u^n + \Delta t f_1(t_{n+1}), \tag{3.4}$$

$$u^{n+\frac{1}{2}}(0, y) = u^{n+\frac{1}{2}}(1, y) = 0, \tag{3.5}$$

$$(I + \Delta t L_{y,\varepsilon})u^{n+1} = u^{n+\frac{1}{2}} + \Delta t f_2(t_{n+1}), \tag{3.6}$$

$$u^{n+1}(x, 0) = u^{n+1}(x, 1) = 0. \tag{3.7}$$

This method gives approximations $u^n(x, y)$ to the solution $u(x, y, t)$ of (1.1) at the time levels $t_n = n\Delta t$. The operators $(I+\Delta t L_{i,\varepsilon}),\ i = x, y$ satisfy a maximum principle, which ensures the stability of the scheme (3.3)–(3.7).

The local truncation error is defined as $e_{n+1} \equiv u(t_{n+1}) - \overline{u}^{n+1}$, where $\overline{u}^{n+1}$ is the solution of

$$(I + \Delta t L_{x,\varepsilon})\overline{u}^{n+\frac{1}{2}}(x, y) = u(x, y, t_n) + \Delta t f_1(x, y, t_{n+1}), \tag{3.8}$$

$$\overline{u}^{n+\frac{1}{2}}(0, y) = \overline{u}^{n+\frac{1}{2}}(1, y) = 0, \tag{3.9}$$

$$(I + \Delta t L_{y,\varepsilon})\overline{u}^{n+1} = \overline{u}^{n+\frac{1}{2}} + \Delta t f_2(x, y, t_{n+1}), \tag{3.10}$$

$$\overline{u}^{n+1}(x, 0) = \overline{u}^{n+1}(x, 1) = 0. \tag{3.11}$$

The following consistency result is obtained.

LEMMA 1 The local error for the scheme (3.8)–(3.11) satisfies

$$\|e_{n+1}\|_\infty \leqslant C(\Delta t)^2. \tag{3.12}$$

Proof. The function $\overline{u}^{n+1}$ satisfies

$$(I + \Delta t L_{x,\varepsilon})\Big((I + \Delta t L_{y,\varepsilon})\overline{u}^{n+1} - \Delta t f_2(t_{n+1})\Big) - \Delta t f_1(t_{n+1}) = u(t_n).$$

On the other hand, since the solution of (1.1) is smooth enough, we have

$$\begin{aligned} u(t_n) &= u(t_{n+1}) + \Delta t[(L_{x,\varepsilon} + L_{y,\varepsilon})u(t_{n+1}) - (f_1(t_{n+1}) + f_2(t_{n+1}))] \\ &\quad + \int_{t_{n+1}}^{t_n} (t_n - s)\frac{\partial^2 u}{\partial t^2}(s)\,\mathrm{d}s \\ &= (I + \Delta t L_{x,\varepsilon})\left[(I + \Delta t L_{y,\varepsilon})u(t_{n+1}) - \Delta t f_2(t_{n+1})\right] \\ &\quad - \Delta t f_1(t_{n+1}) + \mathrm{O}(\Delta t^2). \end{aligned}$$

Hence, e_{n+1} satisfies

$$\begin{cases} (I + \Delta t L_{x,\varepsilon})(I + \Delta t L_{y,\varepsilon})e_{n+1} = \mathrm{O}\,(\Delta t^2), \\ e_{n+1}(0, y) = e_{n+1}(1, y) = e_{n+1}(x, 0) = e_{n+1}(x, 1) = 0. \end{cases}$$

The application of the maximum principle for the operators $I + \Delta t L_{i,\varepsilon},\ i = x, y$, proves (3.12). □

To show the uniform convergence of (3.3)–(3.7), we introduce the global error

$$E_{\Delta t} = \sup_{n \leqslant T/\Delta t} \|u(t_n) - u^n\|_\infty, \tag{3.13}$$

and the following result is obtained.

LEMMA 2 The global error satisfies $E_{\Delta t} \leqslant C\Delta t$. Therefore, the time semidiscretization process is uniformly convergent of order 1.

Proof. See Clavero *et al.* (1993). □

For the analysis of the uniform convergence of the total discretization, we have to study the behaviour, with respect to the singular perturbation parameter ε, of the solutions of problems (3.8)–(3.9) and (3.10)–(3.11).

LEMMA 3 Let $\overline{u}^{n+\frac{1}{2}}(x, y)$ be the solution of problem (3.8)–(3.9). Then, there exists C such that

$$\left|\frac{\partial^i \overline{u}^{n+\frac{1}{2}}(x, y)}{\partial x^i}\right| \leqslant C\left[1 + \varepsilon^{-i/2}\left(\exp\left(-\frac{\gamma x}{\sqrt{\varepsilon}}\right) + \exp\left(-\frac{\gamma(1-x)}{\sqrt{\varepsilon}}\right)\right)\right],$$
$$(x, y) \in \overline{\Omega},\ 0 \leqslant i \leqslant 4. \tag{3.14}$$

Proof. First, using the maximum principle for problem (3.8)–(3.9), where $y \in [0, 1]$ acts as a parameter, we deduce that $|\overline{u}^{n+\frac{1}{2}}(x, y)| \leqslant C$. To prove (3.14) for $i = 1$, we consider the boundary value problem

$$\begin{cases} (I + \Delta t L_{x,\varepsilon})w = -L_{x,\varepsilon}u(x, y, t_n) + f_1(x, y, t_{n+1}), \\ w(0, y) = w(1, y) = 0, \end{cases} \tag{3.15}$$

whose solution is given by

$$w(x, y) = \frac{\overline{u}^{n+\frac{1}{2}}(x, y) - u(x, y, t_n)}{\Delta t}.$$

Taking into account that $|L_{x,\varepsilon}u(x, y, t_n)| \leqslant C$ in $\overline{\Omega}$, the maximum principle implies that $|w| \leqslant C$. Writing now the problem (3.8)–(3.9) in the form

$$\begin{cases} L_{x,\varepsilon}\overline{u}^{n+\frac{1}{2}} = -w + f_1(x, y, t_{n+1}), \\ \overline{u}^{n+\frac{1}{2}}(0, y) = \overline{u}^{n+\frac{1}{2}}(1, y) = 0, \end{cases} \tag{3.16}$$

and proceeding in a similar way as in Clavero (1989) and Miller *et al.* (1996), for the study of one-dimensional stationary problems, we can deduce

$$\left|\frac{\partial^i \overline{u}^{n+\frac{1}{2}}}{\partial x^i}(0, y)\right| \leqslant C\varepsilon^{-1/2}, \qquad i = 1, 2, \quad y \in [0, 1]$$

$$\left|\frac{\partial^i \overline{u}^{n+\frac{1}{2}}}{\partial x^i}(1, y)\right| \leqslant C\varepsilon^{-1/2}, \qquad i = 1, 2, \quad y \in [0, 1].$$

To prove the bound for $(x, y) \in (0, 1) \times [0, 1]$, we differentiate equation (3.8) with respect to x. Then, the function $\partial \overline{u}^{n+\frac{1}{2}}/\partial x$ satisfies

$$(I+\Delta t L_{x,\varepsilon})\frac{\partial \overline{u}^{n+\frac{1}{2}}}{\partial x} = -\frac{\partial u(x, y, t_n)}{\partial x} + \Delta t\frac{\partial f_1(x, y, t_{n+1})}{\partial x} - \frac{\partial k_1}{\partial x}\overline{u}^{n+\frac{1}{2}} \equiv h(x, y). \quad (3.17)$$

From (2.5), (2.9)–(2.16), it follows that

$$|h(x, y)| \leqslant C\left[1 + \varepsilon^{-1/2}\left(\exp\left(-\frac{\gamma x}{\sqrt{\varepsilon}}\right) + \exp\left(-\frac{\gamma(1-x)}{\sqrt{\varepsilon}}\right)\right)\right]. \quad (3.18)$$

Considering the barrier functions

$$\begin{aligned} s_1(x) &= 1 + x, \\ s_2(x) &= \varepsilon^{-\frac{1}{2}}\left[\exp\left(-\frac{\gamma x}{\sqrt{\varepsilon}}\right) + \exp\left(-\frac{\gamma(1-x)}{\sqrt{\varepsilon}}\right)\right], \end{aligned}$$

we can find sufficiently large ε-independent constants C_1, C_2 such that

$$\begin{aligned} &(I + \Delta t L_{x,\varepsilon})\left(C_1 s_1 + C_2 s_2 \pm \frac{\partial \overline{u}^{n+\frac{1}{2}}}{\partial x}\right) \geqslant 0, \\ &C_1 s_1(0) + C_2 s_2(0) \geqslant \left|\frac{\partial \overline{u}^{n+\frac{1}{2}}}{\partial x}(0, y)\right|, \\ &C_1 s_1(1) + C_2 s_2(1) \geqslant \left|\frac{\partial \overline{u}^{n+\frac{1}{2}}}{\partial x}(1, y)\right|. \end{aligned}$$

Then, from the maximum principle

$$\left|\frac{\partial \overline{u}^{n+\frac{1}{2}}}{\partial x}\right| \leqslant C_1 s_1 + C_2 s_2 \leqslant C\left[1 + \varepsilon^{-\frac{1}{2}}\left(\exp\left(-\frac{\gamma x}{\sqrt{\varepsilon}}\right) + \exp\left(-\frac{\gamma(1-x)}{\sqrt{\varepsilon}}\right)\right)\right], \quad (3.19)$$

which is the required result. Similarly we can prove (3.14) for $i = 2$.

To obtain the bounds for the third and fourth derivatives of $\overline{u}^{n+\frac{1}{2}}$, we proceed as follows. Let $\overline{w}(x, y) = L_{x,\varepsilon} w$ be the solution of

$$\begin{cases} (I + \Delta t L_{x,\varepsilon})\overline{w} = -L^2_{x,\varepsilon} u(x, y, t_n) + L_{x,\varepsilon} f_1(x, y, t_{n+1}), \\ \overline{w}(0, y) = \frac{1}{\Delta t}(f_1(0, y, t_{n+1}) - L_{x,\varepsilon} u(0, y, t_n)), \\ \overline{w}(1, y) = \frac{1}{\Delta t}(f_1(1, y, t_{n+1}) - L_{x,\varepsilon} u(1, y, t_n)). \end{cases} \quad (3.20)$$

Then, by (3.2) it is clear that

$$L_{x,\varepsilon} u(0, y, t_n) = f(0, y, t_n) = f_1(0, y, t_n)$$

and also

$$L_{x,\varepsilon} u(1, y, t_n) = f(1, y, t_n) = f_1(1, y, t_n).$$

Thus, we have $|\overline{w}(0, y)| \leqslant C$, $|\overline{w}(1, y)| \leqslant C$, and therefore $|\overline{w}| \leqslant C$ (note that the bounds for the solution of (1.1) detailed in the previous section ensure that $|L^2_{x,\varepsilon} u(x, y, t_n)| \leqslant C$). If we consider the problem

$$\begin{cases} L_{x,\varepsilon} w = \overline{w}, \\ w(0) = w(1) = 0, \end{cases}$$

the bound $|\overline{w}| \leqslant C$ clearly gives

$$\frac{1}{\Delta t}\left|\frac{\partial^2 \overline{u}^{n+\frac{1}{2}}(x, y)}{\partial x^2} - \frac{\partial^2 u(x, y, t_n)}{\partial x^2}\right| \leqslant \frac{C}{\varepsilon}, \qquad \forall\,(x, y) \in \overline{\Omega}. \tag{3.21}$$

Now, using the auxiliary function

$$w_1(x, y) \equiv \frac{\partial^2 \overline{u}^{n+\frac{1}{2}}(x, y)}{\partial x^2},$$

which is the solution of

$$\begin{cases} L_{x,\varepsilon} w_1 = -\dfrac{\partial^2 w}{\partial x^2} - 2\dfrac{\partial k_1}{\partial x}\dfrac{\partial \overline{u}^{n+\frac{1}{2}}}{\partial x} + \dfrac{\partial^2 f_1}{\partial x^2} - \dfrac{\partial k_1^2}{\partial x^2}\overline{u}^{n+\frac{1}{2}}, \\ w_1(0, y) = \dfrac{f_1(0, y, t_{n+1})}{\varepsilon}, \qquad w_1(1, y) = \dfrac{f_1(1, y, t_{n+1})}{\varepsilon}, \end{cases} \tag{3.22}$$

in the same way as for problem (3.16), we can deduce

$$\left|\frac{\partial^i \overline{u}^{n+\frac{1}{2}}}{\partial x^i}(0, y)\right| \leqslant C\varepsilon^{-i/2}, \qquad i = 3, 4,$$

$$\left|\frac{\partial^i \overline{u}^{n+\frac{1}{2}}}{\partial x^i}(1, y)\right| \leqslant C\varepsilon^{-i/2}, \qquad i = 3, 4,$$

from which (3.14) follows for $i = 3, 4$. □

Similar techniques applied to the problem (3.10)–(3.11) give the following result.

LEMMA 4 Let $\overline{u}^{n+1}(x, y)$ be the solution of problem (3.10)–(3.11). Then, there exists C such that

$$\left|\frac{\partial^i \overline{u}^{n+1}(x, y)}{\partial y^i}\right| \leqslant C\left[1 + \varepsilon^{-i/2}\left(\exp\left(-\frac{\gamma y}{\sqrt{\varepsilon}}\right) + \exp\left(-\frac{\gamma(1-y)}{\sqrt{\varepsilon}}\right)\right)\right],$$
$$(x, y) \in \overline{\Omega},\ 0 \leqslant i \leqslant 4. \tag{3.23}$$

4. The spatial discretization

In this section we study the totally discrete scheme obtained from the spatial discretization of (3.8)–(3.11). Let us introduce a nonuniform rectangular mesh $\overline{\Omega}_{\varepsilon,h}$ as the tensor product

$I_{x,\varepsilon,h} \times I_{y,\varepsilon,h}$ of one-dimensional special meshes, generated as follows. To define $I_{x,\varepsilon,h}$ (and likewise for $I_{y,\varepsilon,h}$), let $h = 1/N$ with N such that $N/4 \in \mathbb{N}$ and

$$\sigma = \min(\tfrac{1}{4}, m\sqrt{\varepsilon}\log N), \tag{4.1}$$

where m is a constant independent of ε and N, satisfying $m\gamma \geqslant 1$. Dividing the interval $[0, 1]$ into three subintervals $[0, \sigma]$, $[\sigma, 1-\sigma]$ and $[1-\sigma, 1]$, we define the mesh

$$I_{x,\varepsilon,h} \equiv \{0 = x_0, x_1, \ldots, x_{\frac{N}{4}} = \sigma, \ldots, x_{\frac{3N}{4}} = 1-\sigma, \ldots, x_N = 1\}, \tag{4.2}$$

where $x_i = i\frac{4\sigma}{N}, i = 0, \ldots, \frac{N}{4}$, $x_i = \sigma + (i - \frac{N}{4})\frac{2(1-2\sigma)}{N}, i = \frac{N}{4}+1, \ldots, \frac{3N}{4}$ and $x_i = 1-\sigma+(i-\frac{3N}{4})\frac{4\sigma}{N}, i = \frac{3N}{4}+1, \ldots, N$. Let $h_i = x_i - x_{i-1},\ \rho_i = h_i/\sqrt{\varepsilon},\ i = 1, \ldots, N$ and $\overline{h}_i = (h_i + h_{i+1})/2,\ i = 1, \ldots, N-1$.

Let $[\cdot]_h$ be the operator of restriction to $\overline{\Omega}_{\varepsilon,h}$ of functions defined on Ω. Then the totally discrete approximations $u_h^n = (u_{i,j}^n)$ to $[u(t_n)]_h$ are defined by

$$u_h^0 = [u_0]_h, \tag{4.3}$$

$$(I + \Delta t L_{x,\varepsilon,h})u_h^{n+\frac{1}{2}} = u_h^n + \Delta t[f_1(x, y, t_{n+1})]_h, \tag{4.4}$$

$$u_h^{n+\frac{1}{2}}(0, y) = u_h^{n+\frac{1}{2}}(1, y) = 0, \qquad y \in I_{y,\varepsilon,h}\backslash\{0, 1\}, \tag{4.5}$$

$$(I + \Delta t L_{y,\varepsilon,h})u_h^{n+1} = u_h^{n+\frac{1}{2}} + \Delta t[f_2(x, y, t_{n+1})]_h, \tag{4.6}$$

$$u_h^{n+1}(x, 0) = u_h^{n+1}(x, 1) = 0, \qquad x \in I_{x,\varepsilon,h}\backslash\{0, 1\}. \tag{4.7}$$

Here $L_{x,\varepsilon,h}$ (and similarly $L_{y,\varepsilon,h}$) is the discretization of the differential operator $L_{x,\varepsilon}$ ($L_{y,\varepsilon}$) using the one-dimensional central finite difference scheme on $I_{x,\varepsilon,h}$ ($I_{y,\varepsilon,h}$), i.e., for each $y_j \in I_{y,\varepsilon,h}$ we have

$$(I + \Delta t L_{x,\varepsilon,h})u_{i,j}^{n+\frac{1}{2}} \equiv r_{i,j}^- u_{i-1,j}^{n+\frac{1}{2}} + r_{i,j}^c u_{i,j}^{n+\frac{1}{2}} + r_{i,j}^+ u_{i+1,j}^{n+\frac{1}{2}} = \overline{f}_{i,j}, \quad i = 1, \ldots, N-1, \tag{4.8}$$

$$u_{0,j}^{n+\frac{1}{2}} = 0, \qquad u_{N,j}^{n+\frac{1}{2}} = 0, \tag{4.9}$$

$$r_{i,j}^- = \frac{-\varepsilon\Delta t}{h_i\overline{h}_i}, \qquad r_{i,j}^+ = \frac{-\varepsilon\Delta t}{h_{i+1}\overline{h}_i}, \qquad r_{i,j}^c = 1 + \Delta t k_{1,i,j} - r_{i,j}^- - r_{i,j}^+, \tag{4.10}$$

$$k_{1,i,j} = k_1(x_i, y_j), \qquad \overline{f}_{i,j} = u_{i,j}^n + \Delta t f_1(x_i, y_j, t_{n+1}). \tag{4.11}$$

LEMMA 5 Let $\overline{u}^{n+\frac{1}{2}}$ be the solution of (3.8)–(3.9) and $\overline{u}_h^{n+\frac{1}{2}}$ the solution of the scheme (4.8)–(4.11) taking $u_h^n \equiv u(x_j, y_j, t_n)$. Then,

$$\|[\overline{u}^{n+1/2}]_h - \overline{u}_h^{n+1/2}\|_\infty \leqslant C\,\Delta t\,N^{-1}. \tag{4.12}$$

Therefore the method is uniformly convergent with respect to ε.

Proof. We only analyse the case $\sigma = m\sqrt{\varepsilon}\log N$ (otherwise the problem can be studied in the classical way). We study the local error depending on the position of the point x_i in the mesh. Three typical cases have to be considered.

The local error in (x_i, y_j) is given by

$$\tau_i = (I + \Delta t L_{x,\varepsilon,h})(\overline{u}^{n+\frac{1}{2}}(x_i)) - ((I + \Delta t L_{x,\varepsilon})\overline{u}^{n+\frac{1}{2}})(x_i)$$

for $i = 1, \ldots, N-1$, where the dependence on the parameter y_j is omitted. Consider the following well-known expressions of the remainder of Taylor's formula:

$$R_n(a, p, g) = g^{(n+1)}(\varphi)\frac{(p-a)^{n+1}}{(n+1)!} \text{ or } R_n(a, p, g) = \frac{1}{n!}\int_a^p (p-s)^n g^{(n+1)}(s)\, \mathrm{d}s. \tag{4.13}$$

Case 1. $0 < x_i < \sigma$.

Using Taylor expansions, it is straightforward to show that

$$\tau_i = r_i^- R_3(x_i, x_{i-1}, \overline{u}^{n+\frac{1}{2}}) + r_i^+ R_3(x_i, x_{i+1}, \overline{u}^{n+\frac{1}{2}}). \tag{4.14}$$

Now, using (3.14) and (4.13), we deduce

$$\begin{aligned} |r_i^- R_3(x_i, x_{i-1}, \overline{u}^{n+\frac{1}{2}})| &\leqslant C\Delta t\varepsilon h_i^2(1 + \varepsilon^{-2}(\mathrm{e}^{-\gamma x_{i-1}/\sqrt{\varepsilon}} + \mathrm{e}^{-\gamma(1-x_i)/\sqrt{\varepsilon}})) \\ &\leqslant C\Delta t h_i^2(1 + \varepsilon^{-1}(\mathrm{e}^{-\gamma x_{i-1}/\sqrt{\varepsilon}} + \mathrm{e}^{-\gamma(1-\sigma)/\sqrt{\varepsilon}})) \\ &\leqslant C\Delta t h_i^2 \varepsilon^{-1} \leqslant C\Delta t \frac{\log^2 N}{N^2}, \end{aligned}$$

since $h_i = 4\sigma/N$. Similarly, $|r_i^+ R_3(x_i, x_{i+1}, \overline{u}^{n+\frac{1}{2}})| \leqslant C(\Delta t \log^2 N)/N^2$. Therefore, the local error satisfies

$$|\tau_i| \leqslant C\Delta t N^{-2}\log^2 N. \tag{4.15}$$

Case 2. $\sigma < x_i < 1 - \sigma$.

In this case, we distinguish two situations depending on the value of ρ_i.

(i) If $\rho_i \leqslant 1$, then we proceed as in Case 1 to prove

$$\begin{aligned} |r_i^- R_3(x_i, x_{i-1}, \overline{u}^{n+\frac{1}{2}})| &\leqslant C\Delta t h_i^2(1 + \varepsilon^{-1}(\mathrm{e}^{-\gamma x_{i-1}/\sqrt{\varepsilon}} + \mathrm{e}^{-\gamma(1-x_i)/\sqrt{\varepsilon}})) \\ &\leqslant C\Delta t h_i^2(1 + \varepsilon^{-1}(\mathrm{e}^{-\gamma\sigma/\sqrt{\varepsilon}} + \mathrm{e}^{-\gamma(\sigma+h_i)/\sqrt{\varepsilon}})), \end{aligned}$$

since $x_{i-1} \geqslant \sigma$ and $1 - x_i \geqslant \sigma + h_i$. Given that $\sigma = m\sqrt{\varepsilon}\log N$, we deduce

$$|r_i^- R_3(x_i, x_{i-1}, \overline{u}^{n+\frac{1}{2}})| \leqslant C\Delta t h_i^2 + C\Delta t/N^{m\gamma}.$$

Hence, since $m\gamma \geqslant 1$, we have $|r_i^- R_3(x_i, x_{i-1}, \overline{u}^{n+\frac{1}{2}})| \leqslant C\Delta t/N$. In the same way, $|r_i^+ R_3(x_i, x_{i+1}, \overline{u}^{n+\frac{1}{2}})| \leqslant C\Delta t/N$. Consequently,

$$|\tau_i| \leqslant C\Delta t N^{-1}. \tag{4.16}$$

(ii) If $\rho_i \geqslant 1$, we write the local error in the form

$$\tau_i = r_i^- R_2(x_i, x_{i-1}, \overline{u}^{n+\frac{1}{2}}) + r_i^+ R_2(x_i, x_{i+1}, \overline{u}^{n+\frac{1}{2}}). \tag{4.17}$$

Using (4.13) in the integral form and (3.14), it is easy to obtain

$$\begin{aligned}|r_i^- R_2(x_i, x_{i-1}, \overline{u}^{n+\frac{1}{2}})| &\leqslant C\Delta t\varepsilon \int_{x_{i-1}}^{x_i} (1 + \varepsilon^{-3/2}(\mathrm{e}^{-\gamma s/\sqrt{\varepsilon}} + \mathrm{e}^{-\gamma(1-s)/\sqrt{\varepsilon}}))\,\mathrm{d}s \\ &\leqslant C\Delta t\varepsilon(h_i + \varepsilon^{-1}(\mathrm{e}^{-\gamma x_{i-1}/\sqrt{\varepsilon}} + \mathrm{e}^{-\gamma(1-x_i)/\sqrt{\varepsilon}})) \\ &\leqslant C\Delta t\left(h_i + \frac{1}{N^{m\gamma}}\right).\end{aligned}$$

As $m\gamma \geqslant 1$, we have $|r_i^- R_2(x_i, x_{i-1}, \overline{u}^{n+\frac{1}{2}})| \leqslant C\Delta t/N$. Similarly, we can prove that $|r_i^+ R_2(x_i, x_{i+1}, \overline{u}^{n+\frac{1}{2}})| \leqslant C\Delta t/N$. Using the last two estimates, we deduce

$$|\tau_i| \leqslant C\Delta t N^{-1}. \tag{4.18}$$

Case 3. $x_i = \sigma$.

In this case, we again use (4.17) for the local error and distinguish two situations depending on ρ_{i+1}.

(i) If $\rho_{i+1} \leqslant 1$, as in Case 2(i) we have

$$|\tau_i| \leqslant C\Delta t\left(\frac{\varepsilon}{N} + \frac{\log N}{N N^{m\gamma}} + \frac{1}{N^{m\gamma}}\right) \leqslant C\Delta t N^{-1}. \tag{4.19}$$

(ii) If $\rho_{i+1} \geqslant 1$, then we deduce, as in Case 2(ii),

$$\begin{aligned}|r_i^- R_2(x_i, x_{i-1}, \overline{u}^{n+\frac{1}{2}})| &\leqslant C\Delta t\left(\varepsilon h_i + \frac{1}{N^{m\gamma}}\right), \\ |r_i^+ R_2(x_i, x_{i+1}, \overline{u}^{n+\frac{1}{2}})| &\leqslant C\Delta t\left(\varepsilon h_i + \mathrm{e}^{-\gamma x_i/\sqrt{\varepsilon}} + \mathrm{e}^{-\gamma(1-x_{i+1})/\sqrt{\varepsilon}}\right) \\ &\leqslant C\Delta t\left(\varepsilon h_i + \frac{1}{N^{m\gamma}}\right),\end{aligned}$$

taking into account that $1 \geqslant 2\sigma + 2h_{i+1}$. Finally, since $m\gamma \geqslant 1$ we obtain

$$|\tau_i| \leqslant C\Delta t N^{-1}. \tag{4.20}$$

The uniform consistency follows from (4.15), (4.16), (4.18), (4.19), (4.20). As the operator $I + \Delta t L_{x,\varepsilon,h}$ satisfies the discrete maximum principle, we have

$$\|(I + \Delta t L_{x,\varepsilon,h})^{-1}\|_\infty \leqslant \frac{1}{1 + \gamma^2 \Delta t}. \tag{4.21}$$

Hence, the scheme (4.8)–(4.11) is uniformly stable and therefore uniformly convergent. $\square$

THEOREM 6 Let $\overline{u}^{n+1}$ be the solution of (3.8)–(3.11) and $\overline{u}_h^{n+1}$ the numerical solution of (4.3)–(4.7) taking $u_h^n \equiv [u(t_n)]_h$. Then

$$\|[\overline{u}^{n+1}]_h - \overline{u}_h^{n+1}\|_\infty \leqslant C\,\Delta t\, N^{-1}. \tag{4.22}$$

Proof. In the first half step of the algorithm (3.4)–(3.7), we have the one-dimensional stationary singularly perturbed problems (3.8)–(3.9) and their discrete versions (4.8)–(4.9). Then, from Lemma 5 we have

$$\|[\overline{u}^{n+\frac{1}{2}}]_h - \overline{u}_h^{n+\frac{1}{2}}\|_\infty \leqslant C\,\Delta t\, N^{-1}.$$

In the second half step, we have problem (3.10)–(3.11), whose discretization is

$$\begin{aligned} &(I + \Delta t L_{y,\varepsilon,h})\overline{u}_h^{n+1} = \overline{u}_h^{n+1/2} + \Delta t[f_2(x, y, t_{n+1})]_h, \\ &\overline{u}_h^{n+1}(x, 0) = \overline{u}_h^{n+1}(x, 1) = 0, \qquad x \in I_{x,\varepsilon,h}. \end{aligned} \tag{4.23}$$

In order to find the relation between $\overline{u}^{n+1}$ and $\overline{u}_h^{n+1}$, we introduce the auxiliary problem

$$\begin{aligned} &(I + \Delta t L_{y,\varepsilon,h})\tilde{u}_h^{n+1} = [\overline{u}^{n+\frac{1}{2}}]_h + \Delta t[f_2(t_{n+1})]_h, \\ &\tilde{u}_h^{n+1}(x, 0) = \tilde{u}_h^{n+1}(x, 1) = 0, \qquad x \in I_{x,\varepsilon,h}. \end{aligned} \tag{4.24}$$

With the same arguments as in Lemma 5, we can prove

$$\|[\overline{u}^{n+1}]_h - \tilde{u}_h^{n+1}\|_\infty \leqslant C\,\Delta t\, N^{-1}.$$

Writing

$$[\overline{u}^{n+1}]_h - \overline{u}_h^{n+1} = [\overline{u}^{n+1}]_h - \tilde{u}_h^{n+1} + \tilde{u}_h^{n+1} - \overline{u}_h^{n+1},$$

noting that

$$\tilde{u}_h^{n+1} - \overline{u}_h^{n+1} = (I + \Delta t L_{y,\varepsilon,h})^{-1}([\overline{u}^{n+\frac{1}{2}}]_h - \overline{u}_h^{n+\frac{1}{2}}),$$

and since

$$\|(I + \Delta t L_{y,\varepsilon,h})^{-1}\|_\infty \leqslant \frac{1}{1 + \gamma^2 \Delta t}, \tag{4.25}$$

the bound (4.22) follows. □

THEOREM 7 Let u be the solution of (1.1) and $\{u_h^n\}_n$ the solution of (4.3)–(4.7). Then, there exists a constant C such that

$$\|[u(t_n)]_h - u_h^n\|_\infty \leqslant C(\Delta t + N^{-1}). \tag{4.26}$$

Proof. To prove the uniform convergence of the totally discrete scheme we bound the error in the form

$$\|[u(t_n)]_h - u_h^n\|_\infty \leqslant \|[u(t_n)]_h - [\overline{u}^n]_h\|_\infty + \|[\overline{u}^n]_h - \overline{u}_h^n\|_\infty + \|\overline{u}_h^n - u_h^n\|_\infty. \tag{4.27}$$

Then, using the results of Lemma 1, Theorem 6 and (4.21), (4.25) we have

$$\|[u(t_n)]_h - u_h^n\|_\infty \leqslant C((\Delta t)^2 + \Delta t N^{-1}) + \frac{\|[u(t_{n-1})]_h - u_h^{n-1}\|_\infty}{(1 + \gamma^2 \Delta t)(1 + \gamma^2 \Delta t)}. \tag{4.28}$$

Finally, by recurrent applications of (4.28), (4.26) follows. □

TABLE 1
Maximum nodal erors $E_{\varepsilon,N,\Delta t}$ and $E_{N,\Delta t}$

ε	$N = 8$ $\Delta t = 0{\cdot}1$	$N = 16$ $\Delta t = 0{\cdot}05$	$N = 32$ $\Delta t = 0{\cdot}025$	$N = 64$ $\Delta t = 0{\cdot}0125$	$N = 128$ $\Delta t = 0{\cdot}00625$
1	2·622E−4	1·469E−4	8·119E−5	4·598E−5	2·269E−5
10^{-1}	2·405E−3	1·097E−3	5·381E−4	2·795E−4	1·424E−4
10^{-2}	5·757E−3	2·096E−3	1·101E−3	5·671E−4	2·874E−4
10^{-3}	1·594E−2	6·492E−3	2·088E−3	6·564E−4	3·221E−4
10^{-4}	2·052E−2	1·005E−2	4·441E−3	1·713E−3	5·154E−4
10^{-5}	2·209E−2	1·141E−2	5·588E−3	2·622E−3	1·087E−3
10^{-6}	2·260E−2	1·186E−2	5·999E−3	3·003E−3	1·410E−3
10^{-7}	2·276E−2	1·201E−2	6·133E−3	3·133E−3	1·532E−3
10^{-8}	2·281E−2	1·205E−2	6·176E−3	3·176E−3	1·573E−3
$E_{N,\Delta t}$	2·281E−2	1·205E−2	6·176E−3	3·176E−3	1·573E−3

TABLE 2
Numerical order of convergence

ε	$N = 8$	$N = 16$	$N = 32$	$N = 64$
1	0·835	0·855	0·820	1·018
10^{-1}	1·132	1·027	0·945	0·972
10^{-2}	1·457	0·928	0·957	0·980
10^{-3}	1·295	1·636	1·669	1·029
10^{-4}	1·029	1·178	1·374	1·732
10^{-5}	0·953	1·029	1·091	1·270
10^{-6}	0·935	0·983	0·998	1·090
10^{-7}	0·922	0·969	0·969	1·032
10^{-8}	0·920	0·964	0·959	1·013
Unif.	0·920	0·964	0·959	1·013

5. Numerical results

In this section we show numerical results obtained with the scheme (4.3)–(4.11) in the integration of two problems of type (1.1). We have considered an example whose exact solution is known in order to compute exactly the pointwise errors $e_{\varepsilon,N,\Delta t}(x_i, y_j, t_n) = |u(x_i, y_j, t_n) - u^N(x_i, y_j, t_n)|$, where $u^N(x_i, y_j, t_n)$ indicates the approximate solution obtained on a mesh using $N + 1$ points in each spatial direction, and $t_n = n\Delta t$ with

Δt the (constant) time step. For each ε the maximum nodal error is given by

$$E_{\varepsilon,N,\Delta t} = \max_{i,j,n} e_{\varepsilon,N,\Delta t}(x_i, y_j, t_n), \tag{5.1}$$

and for each N and Δt, the ε-uniform maximum nodal error is defined by $E_{N,\Delta t} = \max_\varepsilon E_{\varepsilon,N,\Delta t}$. Computed values of $E_{\varepsilon,N,\Delta t}$ and $E_{N,\Delta t}$ are given in Table 1 for several values of ε, N and Δt. To obtain the numerical ε-uniform rate of convergence p, we use

$$p = \frac{\log \frac{E_{\varepsilon,N,\Delta t}}{E_{\varepsilon,2N,\Delta t}}}{\log 2}.$$

The results are given in Table 2.

A second example with unknown exact solution is also considered. In this case the pointwise error is estimated by using the double mesh principle, i.e., $e^*_{\varepsilon,N,\Delta t}(x_i, y_j, t_n) = |u^*(x_i, y_j, t_n) - u^N(x_i, y_j, t_n)|$, where u^* is an extension to $\overline{\Omega}$, by using bilinear interpolation, of the numerical solution obtained with $2N+1$ points in each spatial direction and a half-size time step. We define $E^*_{\varepsilon,N,\Delta t}$ and $E^*_{N,\Delta t}$ as before. Computed values of $E^*_{\varepsilon,N,\Delta t}$ and $E^*_{N,\Delta t}$ are given in Table 3 and the numerical ε-uniform rates of convergence are given in Table 4.

In both examples, we choose the following decomposition of the function f:

$$f_1(x, y, t) = f(x, y, t) - f_2(x, y, t),$$

$$f_2(x, y, t) = f(x, 0, t) + y(f(x, 1, t) - f(x, 0, t)).$$

This splitting satisfies (3.2).

Finally, we remark that the decomposition of $k(x, y)$ into two functions k_1 and k_2, satisfying $k_i > 0$, $i = 1, 2$, does not influence the convergence results of the numerical method and only affects the error constant C.

Example 1

$$\begin{aligned}
&u_t - \varepsilon\Delta u + (5 + x^2y^2 + \sin(\pi x)\sin(\pi y))u = f(x, y, \varepsilon, t), \qquad (x, y, t) \in \Omega \times (0, 1],\\
&u(0, y, t) = u(1, y, t) = 0, \qquad y \in [0, 1],\ t \in [0, 1],\\
&u(x, 0, t) = u(x, 1, t) = 0, \qquad x \in [0, 1],\ t \in [0, 1],\\
&u(x, y, 0) = 0, \qquad x, y \in [0, 1],
\end{aligned}$$

where f is such that the exact solution is given by

$$u_\varepsilon(x, y, t) = ((t + 1)\mathrm{e}^{-t} - 1)(h_1(x) - 1 - \mathrm{e}^{-1/\sqrt{\varepsilon}})(h_2(y) - 1 - \mathrm{e}^{-2/\sqrt{\varepsilon}}),$$

with

$$h_1(\zeta) = \frac{\mathrm{e}^{-\zeta/\sqrt{\varepsilon}} + \mathrm{e}^{-(1-\zeta)/\sqrt{\varepsilon}}}{1 + \mathrm{e}^{-1/\sqrt{\varepsilon}}}, \qquad h_2(\zeta) = \frac{\mathrm{e}^{-2\zeta/\sqrt{\varepsilon}} + \mathrm{e}^{-2(1-\zeta)/\sqrt{\varepsilon}}}{1 + \mathrm{e}^{-1/\sqrt{\varepsilon}}}.$$

For this example we take the following decomposition of the reaction term

$$k_1(x, y) = 1 + \frac{x^2y^2}{2} + \frac{\sin(\pi x)\sin(\pi y)}{2}, \qquad k_2(x, y) = 4 + \frac{x^2y^2}{2} + \frac{\sin(\pi x)\sin(\pi y)}{2}.$$

TABLE 3
Maximum nodal errors $E^*_{\varepsilon,N,\Delta t}$ and $E^*_{N,\Delta t}$

ε	$N = 8$ $\Delta t = 0{\cdot}1$	$N = 16$ $\Delta t = 0{\cdot}05$	$N = 32$ $\Delta t = 0{\cdot}025$	$N = 64$ $\Delta t = 0{\cdot}0125$	$N = 128$ $\Delta t = 0{\cdot}00625$
2^{0}	2·251E−3	1·760E−3	1·148E−3	6·669E−4	3·589E−4
2^{-2}	4·066E−3	2·499E−3	1·386E−3	7·321E−4	3·756E−4
2^{-4}	5·225E−3	2·832E−3	1·464E−3	7·427E−4	3·738E−4
2^{-6}	6·473E−3	3·137E−3	1·495E−3	7·272E−4	3·557E−4
2^{-8}	3·945E−3	2·458E−3	1·340E−3	7·302E−4	3·289E−4
2^{-10}	4·342E−3	2·071E−3	1·234E−3	5·544E−4	2·724E−4
2^{-12}	5·792E−3	2·087E−3	1·107E−3	4·666E−4	2·453E−4
2^{-14}	6·593E−3	2·475E−3	1·046E−3	4·545E−4	2·416E−4
2^{-16}	6·994E−3	2·677E−3	1·098E−3	4·978E−4	2·429E−4
2^{-18}	7·191E−3	2·777E−3	1·150E−3	5·246E−4	2·513E−4
2^{-20}	7·288E−3	2·826E−3	1·175E−3	5·379E−4	2·585E−4
$E^*_{N,\Delta t}$	7·288E−3	3·137E−3	1·495E−3	7·427E−4	3·756E−4

TABLE 4
Numerical order of convergence

ε	$N = 8$	$N = 16$	$N = 32$	$N = 64$
2^{0}	0·355	0·616	0·784	0·894
2^{-2}	0·702	0·850	0·921	0·963
2^{-4}	0·884	0·952	0·979	0·990
2^{-6}	1·045	1·069	1·040	1·031
2^{-8}	0·683	0·875	0·876	1·151
2^{-10}	1·068	0·747	1·154	1·025
2^{-12}	1·472	0·915	1·246	0·927
2^{-14}	1·413	1·242	1·204	0·911
2^{-16}	1·385	1·285	1·142	1·035
2^{-18}	1·372	1·272	1·132	1·062
2^{-20}	1·366	1·266	1·127	1·057
Unif.	1·216	1·069	1·009	0·983

The parameters used to generate the special meshes are given by

$$\sigma_x = \min(\tfrac{1}{4},\ \sqrt{\varepsilon}\log N), \qquad \sigma_y = \min(\tfrac{1}{4},\ 0{\cdot}5\sqrt{\varepsilon}\log N).$$

Example 2

$$\begin{aligned}
&u_t - \varepsilon\Delta u + (2+xy)u = f(x, y, \varepsilon, t), \qquad (x, y, t) \in \Omega \times (0, 2],\\
&u(0, y, t) = u(1, y, t) = 0, \qquad y \in [0, 1],\ t \in [0, 2],\\
&u(x, 0, t) = u(x, 1, t) = 0, \qquad x \in [0, 1],\ t \in [0, 2],\\
&u(x, y, 0) = 0, \qquad x, y \in [0, 1],
\end{aligned}$$

where

$$f(x, y, \varepsilon, t) = \left(\max\left\{0, \cos\pi\left((x-0{\cdot}5)^2 + (y-0{\cdot}5)^2\right) - \mathrm{e}^{-t}\right\}\right)^2.$$

For this example we decompose the reaction term in the form

$$k_1(x, y) = 1 + \frac{xy}{2}, \qquad k_2(x, y) = 1 + \frac{xy}{2}.$$

The parameters used to generate the special meshes are

$$\sigma_x = \sigma_y = \min(\tfrac{1}{4},\ \sqrt{\varepsilon}\log N).$$

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